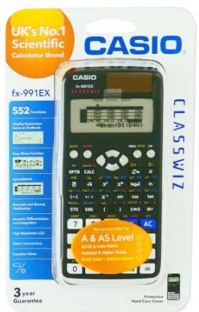
NIHIL NISI DEO

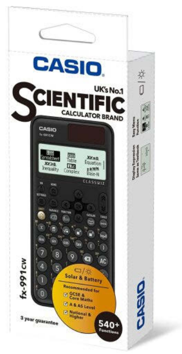
A Level Mathematics Induction Pack

To be accepted onto the A Level Mathematics course, you need to have achieved a minimum of a grade 7 at GCSE Mathematics.

You will need to purchase:

- A Calculator:
    - Casio FX991-CW replaces the Casio FX991-EX. We recommend you get 1 of these 2 versions. These are available on Parent Pay at a discounted rate (approximately £25).
      This will be available to you from September. If you see them cheap over the summer holidays, get one then and there but make sure it's the correct model.





You will be given:

- Digital access to textbooks
- Digital access to revision workbooks

A level mathematics builds on what you already know and introduces new concepts like calculus! Maths is all connected. It’s not simply “algebra” or “number” – it’s a combination of them all.

At A level, you will take these interconnected ideas and use them to model real-life situations. You could use calculus to determine the best placement for a tennis player on a court, mechanics to model the movement of a tennis ball after it leaves the racket, or statistics to model the probability of the tennis player winning the grand slam.

It is essential that you’re able to bring together all the skills you’ve already developed in order to solve problems at A level.

This booklet revises those key skills and encourages you to look at them in a slightly different way. Make sure to complete this booklet over the summer and bring it with you for your first day in September.

A Level Mathematics is split into Pure and Applied. Assessment is at the end of year 13 and is split across 3 papers. Topics and assessment details are in the table below. All papers are equally weighted. You will have 1 teacher for Pure Mathematics and a different teacher for Applied Mathematics split equally to give a total of 8 lessons per fortnight.

Paper	Duration	Topics Covered
Paper 1: Pure Mathematics 1	2 hours 100 marks	• Topic 1 – Proof • Topic 2 – Algebra and functions • Topic 3 – Coordinate geometry in the (x, y) plane • Topic 4 – Sequences and series • Topic 5 – Trigonometry • Topic 6 – Exponentials and logarithms • Topic 7 – Differentiation • Topic 8 – Integration • Topic 9 – Numerical methods • Topic 10 – Vectors
Paper 2: Pure Mathematics 2	2 hours 100 marks	
Paper 3: Statistics and Mechanics	2 hours 100 marks Section A: 50 marks Section B: 50 marks	**Section A: Statistics** • Topic 1 – Statistical sampling • Topic 2 – Data presentation and interpretation • Topic 3 – Probability • Topic 4 – Statistical distributions • Topic 5 – Statistical hypothesis testing **Section B: Mechanics** • Topic 6 – Quantities and units in mechanics • Topic 7 – Kinematics • Topic 8 – Forces and Newton's laws • Topic 9 – Moments

Contents

Number

At A level, there are no non-calculator papers and you are encouraged to use a calculator, where possible, to solve problems (including using calculators that can solve linear and polynomial equations).

You might think this makes a “Number” section pointless, but the tools included here become increasingly important. The line between what is a “Number” problem and what is an “Algebra” problem becomes very blurred, so you will need to be comfortable combining indices, surds and algebra.

At the same time, as the problems you solve become more complex, maintaining accuracy becomes more important. You’ll need to be comfortable working with intermediate answers as surds (or fractions, or in terms of π) to put off rounding until you have a final answer.

Negative and Zero Powers

An index is an instruction telling you how many times to multiply a number by itself, e.g. $5^3 = 5 \times 5 \times 5$. However, that index doesn't have to be a positive whole number; it can also be negative, zero or even a fraction. Look at the pattern of these indices to see how this works:

$$
\begin{array}{cccccc}
 & 2^3 & = & 2 \times 2 \times 2 & \\
\times 2 & & & & & \div 2 \\
 & 2^2 & = & 2 \times 2 & \\
\times 2 & & & & & \div 2 \\
 & 2^1 & = & 2 & \\
\times 2 & & & & & \div 2 \\
 & 2^0 & = & 1 & \\
\times 2 & & & & & \div 2 \\
 & 2^{-1} & = & \frac{1}{2} & \\
\times 2 & & & & & \div 2 \\
 & 2^{-2} & = & \frac{1}{2 \times 2} & \\
\times 2 & & & & & \div 2 \\
 & 2^{-3} & = & \frac{1}{2 \times 2 \times 2} &
\end{array}
$$

The first thing to notice is that $2^0 = 1$. Anything with an index of 0 is 1, whether the base is large, small or algebraic.

Example 1

$0.5^0 =$ **1**

$19\ 251\ 215^0 =$ **1**

$(3x^2 + 8x + 10)^0 =$ **1**

Secondly, when we are dealing with negative indices, we are effectively dividing instead of multiplying.

$2^3 =$ **8**

and $2^{-3} = \frac{1}{2^3} = \mathbf{\frac{1}{8}}$

This means that a negative power tells us to find the reciprocal of the number. Remember, the reciprocal of a number is what we get when we divide 1 by that number:

The reciprocal of x is $\frac{1}{x}$.

If the number is already a fraction, this is the same as inverting the numerator and denominator: The reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$.

Example 2

Evaluate 4^{-3}

The negative tells us to find the reciprocal. The reciprocal of 4 is $\frac{1}{4}$.

$4^{-3} = (\frac{1}{4})^3 = \frac{1}{4^3} = \mathbf{\frac{1}{64}}$

Notice that we are cubing both parts of the fraction, but since 1^3 is 1, we just write 1.

Likewise, if the base is a fraction, we find the reciprocal and then apply the index to both the numerator and denominator.

Example 3

a. Evaluate $(\frac{3}{5})^{-2}$

The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

$(\frac{3}{5})^{-2} = \frac{5^2}{3^2} = \mathbf{\frac{25}{9}}$

b. Evaluate $(\frac{1}{2})^{-2}$

$(\frac{1}{2})^{-2} = \frac{2^2}{1^2} = \frac{4}{1} = \mathbf{4}$

c. Evaluate $(\frac{2}{5})^{-3}$

$(\frac{2}{5})^{-3} = \frac{5^3}{2^3} = \mathbf{\frac{125}{8}}$

Your Turn:

1\. a. What is $3^5 \div 3^2$ in index form?

____

b. What is $3^2 \div 3^2$ in index form?

____

c. Evaluate $3^2 \div 3^2$

____

d. Evaluate 3^0

____

e. Evaluate 27.54^0

____

f. Evaluate $2.7523^0 \times 268^1 \times 892^0$

____

2\. Evaluate the following:

a. 5^{-2} ________

b. 8^{-2} ________

c. 3^{-3} ________

d. 2^{-5} ________

3. Write each in index form:

a. $\frac{1}{16}$ b. $\frac{1}{49}$ c. $\frac{1}{125}$ d. $\frac{1}{1000}$

______ ______ ______ ______

4. Evaluate, giving your answers as fractions in their simplest form:

a. $(\frac{3}{5})^{-1}$ b. $(\frac{7}{8})^{-2}$ c. $(\frac{1}{4})^{-3}$ d. $(\frac{2}{3})^{-3}$

______ ______ ______ ______

5. Evaluate, giving your answers as fractions in their simplest form:

a. $(3x)^{-2}$ b. $(2x^3)^{-2}$ c. $(5x^4)^{-3}$ d. $(2x^2y^3)^{-4}$

______ ______ ______ ______

For answers, go to page 92.

Fractional Powers

Fractional powers (fractional indices) are a way of representing a combination of roots or powers. The denominator represents the value of the root.

For example, a fractional power of $\frac{1}{2}$ represents a square root and $\frac{1}{3}$ represents a cube root.

Example 1

$25^{\frac{1}{2}} = \sqrt{25} = \mathbf{5}$

$8^{\frac{1}{3}} = \sqrt[3]{8} = \mathbf{2}$

$(4x^2)^{\frac{1}{2}} = \sqrt{4x^2}$
$= \sqrt{4}\sqrt{x^2}$
$= \mathbf{2x}$

The numerator of a fractional power represents an integer power, which allows a fractional power to represent both a power and a root.

For example, a fractional power of $\frac{3}{2}$ would tell you to raise the base to a power of 3, and to square root the answer. These two operations can be done in either order; you can use your judgement to decide which is easier.

Example 2

$4^{\frac{3}{2}}$
$= \sqrt{4^3}$
$= \sqrt{64}$
$= \mathbf{8}$

or $4^{\frac{3}{2}}$
$= (\sqrt{4})^3$
$= 2^3$
$= \mathbf{8}$

As you can see, the answer is the same whether you evaluate the power or the root first.

Example 3

$9^{\frac{3}{2}}$
$= \sqrt{9^3}$
$= \sqrt{729}$
$= \mathbf{27}$

When this is combined with negative powers, you are likely to have three operations to carry out. Again, the order will not change the result:

E.g. $(\frac{9}{25})^{-\frac{3}{2}} = (\frac{25}{9})^{\frac{3}{2}}$
$= (\frac{\sqrt{25}}{\sqrt{9}})^3$
$= \frac{5^3}{3^3}$
$= \mathbf{\frac{125}{9}}$

Your Turn:

1. Evaluate the following:

   a. $36^{\frac{1}{2}}$ b. $1000^{\frac{1}{3}}$ c. $64^{\frac{1}{3}}$ d. $81^{-\frac{1}{2}}$

2. Evaluate the following:

   a. $27^{\frac{2}{3}}$ b. $8^{\frac{4}{3}}$ c. $49^{\frac{3}{2}}$ d. $64^{\frac{2}{3}}$

3. Express in the form $a^{\frac{m}{n}}$, where m and n are integers.

   a. $\sqrt{a^3}$

   c. $\dfrac{1}{\sqrt{a^7}}$

   b. $\sqrt[3]{a^5}$

   d. $\sqrt{a} \times \dfrac{1}{\sqrt{a^5}}$

4. Write the following expressions in order, from smallest to largest:

   $25^{\frac{1}{2}}$, $8^{\frac{2}{3}}$, $27^{\frac{1}{3}}$, $(\frac{1}{9})^{-\frac{3}{2}}$, $(\frac{1}{12})^{-1}$, $(27^{\frac{5}{3}})^0$,

5. Write $64^{\frac{2}{3}} \times 2^3$ in the form 2^a, where a is a positive integer.

For answers, go to page 93.

Index Laws

The laws of indices are shortcuts for simplifying expressions involving indices without evaluating them.

Index Law for Multiplication: $\boldsymbol{a^x \times a^y = a^{x+y}}$

E.g. Simplify $3^5 \times 3^2$. Give your answer in index notation.

Firstly, remember what an index is – an instruction to multiply a number by itself.

$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$3^2 = 3 \times 3$

$3^5 \times 3^2 = \underline{3 \times 3 \times 3 \times 3 \times 3} \times \underline{3 \times 3} = \mathbf{3^7}$

The index law for multiplication lets you skip the intermediate steps and go straight to $3^5 \times 3^2 = 3^7$

It's important to remember that all three of these laws work with both numerical and algebraic bases. Read the question carefully. If it says evaluate, you are expected to give a numerical answer without an index. If it says index notation, you are expected to give it in the form a^b.

E.g.: $x^3 \times x^5 = x^8$

$5^a \times 5^{2b} = \mathbf{5^{a+2b}}$

You can only use these laws if both bases are the same:

E.g.: $3^5 \times 5^2 \neq 15^7$

$a^5 \times b^3 \times a^2 \times b^3 = \boldsymbol{a^7 \times b^6}$

Index Law for Division: $\boldsymbol{a^x \div a^y = a^{x-y}}$

E.g. $a^7 \div a^4$

$$= \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a}$$

$$= \frac{a \times a \times a \times \cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}} \qquad \text{(cancel out where possible)}$$

$$= \boldsymbol{a^3}$$

Or, using the index law as a shortcut: $a^7 \div a^4 = a^{7-4} = \boldsymbol{a^3}$

Again, remember that this will work with both numerical and algebraic terms. Be very careful with negative numbers. It is easy to make mistakes when dealing with negative indices.

E.g.: $5^4 \div 5^{-3} = 5^{4--3} = 5^{4+3} = \mathbf{5^7}$

$x^{-5} \div x^7 = x^{-5-7} = \boldsymbol{x^{-12}}$

Index Law for Powers: $(a^x)^y = a^{x \times y}$

E.g. $(5^3)^4$

$= \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5} \times \underline{5 \times 5 \times 5}$

$= \mathbf{5^{12}}$

Or, using the index law for powers: $(5^3)^4 = 5^{3 \times 4} = 5^{12}$

Again, be careful of negative numbers. Sometimes, you will have to apply more than one of these laws, the order will not change the result. Which order you apply them is up to your judgement.

E.g.: $\left(\frac{(a^4 \times a^3)}{(a^2)}\right)^2$

$= \left(\frac{(a^7)}{(a^2)}\right)^2$ Use the index law for multiplication on the numerator.

$= (a^5)^2$ Then use the index law for division to remove the fraction.

$= \boldsymbol{a^{10}}$ Then use the index law for powers to get a final answer.

Your Turn:

1. Simplify each expression. Give your answers in index form.

   a. $5^4 \times 5^8$ b. $m^4 \div m^2$ c. $(a^3)^2$ d. $3^5 \times 3$

2. Simplify each expression. Give your answers in index form.

   a. $3^8 \times 3^{-2}$ b. $\frac{h^{-3}}{h^5}$ c. $p^{-2} \div p^{-9}$ d. $(5^{-3})^{-2}$

3. Simplify each expression. Give your answers in index form.

   a. $3a^2 \times 3a^5$ b. $(3x^4)^3$ c. $\frac{12x^3}{4x^5}$ d. $a^2b^5 \times a^4b^{-8}$

4. Simplify the expression. Give your answer in index form.

   $\left(\frac{3a^5 \times 6a^{-7}}{2a^5}\right)^2$

For answers, go to page 95.

Simplifying Surds

When you find the square root of an integer, your answer will be one of two types of number. If you find the root of a square number, your result will be an integer: $\sqrt{4} = 2$; if you find the square root of any other integer, your answer will be irrational: $\sqrt{2} = 1.414213562373095...$

A rational number is any number that can be represented as a fraction of two integers, e.g. $\frac{1}{3}$ (even though it recurs) or $\frac{3}{1}$ (the integer 3), while an irrational number is any number that is not rational.

An irrational number will never end and never repeat so if a square root results in an irrational number, you either have to round your answer to write it, or leave it as a root.

If you round it, you have lost some of the accuracy of the answer. A surd is therefore an irrational number that has been left in the form of a root to represent its exact value.

For example, we want to find the value of $5x^2 + 3$, where $x = \sqrt{6}$.

We have two options: 1. Use $x = \sqrt{6}$;

2. Use $x = 2.4$ ($\sqrt{6}$ rounded to 1d.p.)

1. $5(\sqrt{6})^2 + 3 = 5 \times 6 + 3 = \mathbf{33}$
2. $5 \times 2.4^2 + 3 = 5 \times 5.76 + 3 = \mathbf{31.8}$

Even with an example with small numbers, you can see how different the rounded answer is. In any situation dealing with roots, it is therefore important that you can operate with numbers in surd form.

To do this, you need to be able to simplify surds, starting with multiplying and dividing. Here are some examples:

E.g.

a. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$

$= \sqrt{\mathbf{6}}$

b. $2\sqrt{5} \times 4\sqrt{2} = 2 \times 4 \times \sqrt{5 \times 2}$

$= \mathbf{8\sqrt{10}}$

c. $\sqrt{10} \div \sqrt{5} = \sqrt{10 \div 5}$

$= \sqrt{\mathbf{2}}$

d. $\sqrt{10} \div \sqrt{3} = \sqrt{\mathbf{\frac{10}{3}}}$

When multiplying surds, you simply multiply the base of each root.

If there are coefficients outside the root, multiply them separately.

You can divide the bases if the result is an integer. If the result is not an integer, you will normally write the surds as a fraction.

Your Turn:

1. a. $\sqrt{5} \times \sqrt{7}$

   d. $18\sqrt{20} \div 6\sqrt{5}$

   g. $(5\sqrt{3})^2$

   b. $3\sqrt{2} \times 4\sqrt{5}$

   e. $5\sqrt{2} \times 3\sqrt{8}$

   h. $(2\sqrt{5})^3$

   c. $\sqrt{15} \div \sqrt{3}$

   f. $2\sqrt{3} \times 5$

2. A right-angled triangle has a height of $6\sqrt{5}$cm and a base of $7\sqrt{3}$cm. Find its area.

For answers, go to page 96.

Addition and subtraction of surds can be more complex. In the same way you cannot add or subtract fractions with different denominators, you cannot simply add or subtract surds with different bases:

E.g. a. $3\sqrt{5} + 8\sqrt{5} = 11\sqrt{5}$

b. $2\sqrt{5} + 8\sqrt{3} \neq 10\sqrt{8}$

In some cases, such as example b above, the addition is not possible and you would leave your answer as $2\sqrt{5} + 8\sqrt{3}$. In other cases, you can simplify one surd so it has the same base as the other:

E.g. $\sqrt{2} + \sqrt{8}$

Initially, this might look impossible. However, 8 has a square factor (4 × 2). This means:

$\sqrt{8}$

$= \sqrt{4 \times 2}$

$= \sqrt{4}\sqrt{2}$

$= \mathbf{2\sqrt{2}}$

The key to simplification is to find a square factor. Since we know $\sqrt{4} = 2$, we can write $\sqrt{8}$ as $2\sqrt{2}$, giving:

$\sqrt{2} + \sqrt{8}$

$= \sqrt{2} + 2\sqrt{2}$

$= \mathbf{3\sqrt{2}}$

Here's another example:

$3\sqrt{3} + 2\sqrt{12}$

$= 3\sqrt{3} + 2\sqrt{3 \times 4}$ Look for a square factor.

$= 3\sqrt{3} + 2(\sqrt{4}\sqrt{3})$

$= 3\sqrt{3} + 2(2\sqrt{3})$

$= 3\sqrt{3} + 4\sqrt{3}$

$= \mathbf{7\sqrt{3}}$

Your Turn:

1. Simplify these surds (remember: the key is to find a square factor).

   a. $\sqrt{20}$ __________
   b. $\sqrt{48}$ __________
   c. $\sqrt{75}$ __________
   d. $5\sqrt{8}$ __________

2. Answer the following, giving your answers in the form $a\sqrt{b}$

   a. $\sqrt{2} + \sqrt{18}$

   ____

   ____

   b. $\sqrt{50} - \sqrt{200}$

   ____

   ____

c. $4\sqrt{80} + 3\sqrt{45}$

d. $2\sqrt{50} + 5\sqrt{32}$

3. A rectangle has a width of $6\sqrt{75}$m and a height of $2\sqrt{12}$m. What is its perimeter?

Challenge

A right-angled triangle has a base of $2\sqrt{18}$cm and a height of $2\sqrt{32}$cm. Find the perimeter of the triangle.

For answers, go to page 96.

Rationalising the Denominator

A surd is a way of expressing an irrational root without losing accuracy. When a surd is on the denominator of a fraction, this fraction can be simplified by replacing that surd with an integer. This is called "rationalising" the denominator.

Consider this fraction:

$$\frac{1}{\sqrt{2}}$$

As 2 is not a square number, $\sqrt{2}$ is irrational and we want to remove it from the denominator. We can do this by multiplying our fraction by $\frac{\sqrt{2}}{\sqrt{2}}$.

As $\frac{\sqrt{2}}{\sqrt{2}}$ cancels to 1, this multiplication does not change the value of the fraction.

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}}$ Multiply the numerators, multiply the denominators

$= \mathbf{\frac{\sqrt{2}}{2}}$ $(\sqrt{2} \times \sqrt{2} = 2)$

There is now a surd in the numerator but the denominator is a rational number, 2. Here are some more examples:

E.g. a. Rationalise the denominator of $\frac{2}{\sqrt{3}}$

$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \mathbf{\frac{2\sqrt{3}}{3}}$

b. Rationalise the denominator of $\frac{7}{\sqrt{6}}$

$\frac{7}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \mathbf{\frac{7\sqrt{6}}{6}}$

Your Turn:

1. Rationalise the denominator of each fraction.

a. $\frac{5}{\sqrt{2}}$ ____________

c. $\frac{\sqrt{2}}{\sqrt{3}}$ ____________

e. $\frac{2 - \sqrt{3}}{\sqrt{3}}$ ____________

b. $\frac{4}{\sqrt{3}}$ ____________

d. $\frac{3}{2\sqrt{5}}$ ____________

2. Rationalise the denominator of $\frac{\sqrt{5}}{\sqrt{80}}$. Give your answer as a fraction in its simplest form.

3. What is $\frac{2\sqrt{2}}{\sqrt{6}} + \frac{1}{\sqrt{3}}$? Give your answer in its simplest terms.

For answers, go to page 98.

A similar method can be used to rationalise fractions with more complicated denominators, such as:

$$\frac{2}{\sqrt{3}+4}$$

In this case, multiplying numerator and denominator by $\sqrt{3}$ will give us:

$$\frac{2\sqrt{3}}{3+4\sqrt{3}}$$

This has not rationalised the denominator - we need to multiply by something different.

Consider what happens when you expand a pair of brackets in the form $(a + b)(a - b)$:

$(a + b)(a - b)$

$= a^2 + ab - ab - b^2$

$= \boldsymbol{a^2 - b^2}$

In this case, all we are left with is the difference of two squares. If a or b were surds, they would now be rational.

We can apply this technique to rationalising denominators.

$\frac{2}{\sqrt{3}+4} \times \frac{\sqrt{3}-4}{\sqrt{3}-4}$	Multiply numerator and denominator by $\sqrt{3} - 4$. Notice we have changed the sign on the denominator (this is called the conjugate).
$= \frac{2\sqrt{3}-8}{(\sqrt{3}+4)(\sqrt{3}-4)}$	Start by expanding the brackets in the numerator
$= \frac{2\sqrt{3}-8}{\sqrt{3}\sqrt{3}+4\sqrt{3}-4\sqrt{3}-16}$	Then, expand the brackets in the denominator
$= \boldsymbol{\frac{2\sqrt{3}-8}{-13}}$ or $\boldsymbol{\frac{8-2\sqrt{3}}{13}}$	Finally, simplify the denominator. Remember: $\sqrt{3} \times \sqrt{3} = 3$

The surd is now rationalised.

Your Turn:

4. a. $\frac{5}{\sqrt{2}+7}$ b. $\frac{1}{\sqrt{5}-3}$ c. $\frac{1+\sqrt{2}}{\sqrt{3}+2}$

Challenge

Amy is laying tiles in her rectangular bathroom. By the time she has finished, she has used $8m^2$ worth of tiles. She knows the length of one side of the room is $(\sqrt{5} + 2)$m but, unfortunately, she has lost her tape measure. Amy still needs to work out the perimeter of the room. Calculate the perimeter of the room, giving your answer in its simplest form.

For answers, go to page 98.

Algebra

This is the last time it will really make sense to have "Algebra" as a separate section. Once you start A level, everything will involve algebra. You'll have algebra in your fractions, algebra in your surds, your equations will be trigonometric, and your circles will be defined by equations. Even your long division will be algebraic.

This makes it extremely important that you are confident with the algebra you already know. The new content you cover will take for granted that you are able to apply this knowledge in challenging situations you won't have seen before. You'll need to be comfortable with the algebraic tools you have, and be able to spot where you can use them; maybe you can factorise here to simplify this fraction… maybe you can solve this trigonometric equation if you notice that it includes a difference between two squares…

Expanding Polynomials

Polynomials appear throughout the AS and A level course. A polynomial is just an expression consisting of variables (like x or y), numbers and positive integer (whole number) powers. The terms in this expression can be added or subtracted. For example, $3x^2 + 2x - 4$ is a polynomial but $3x^{-4}$ is not.

Expanding (Multiplying) Brackets

You will need to be able to multiply out single, double and triple brackets.

To expand single brackets, multiply everything inside the bracket by the term on the outside.

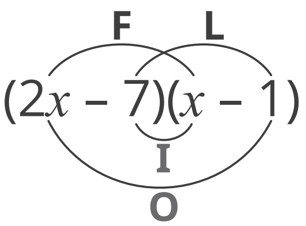
E.g. Expand $3(2x + 5y)$

$3(2x + 5y) = (3 \times 2x) + (3 \times 5y)$

$\qquad = \mathbf{6x + 15y}$

To expand double brackets, multiply everything in the first bracket by everything in the second. The **FOIL** (**f**irst, **o**uter, **i**nner, **l**ast) method can be helpful in ensuring you don't miss any terms.

E.g. Expand $(2x - 7)(x - 1)$



$(2x - 7)(x - 1) = (2x \times x) + (2x \times -1) + (-7 \times x) + (-7 \times -1)$

$\qquad = 2x^2 - 2x - 7x + 7$

$\qquad = \mathbf{2x^2 - 9x + 7}$

To expand triple brackets, begin by multiplying one pair of brackets, then multiply each term in the expanded expression by each term in the remaining bracket. You will need to be systematic in your approach so you don't lose any terms. A grid method can help.

E.g. Expand $(x - 1)(x + 2)(x + 3)$

$(x - 1)(x + 2)(x + 3) = (x^2 + 2x - x - 2)(x + 3)$

$\qquad = (x^2 + x - 2)(x + 3)$

	x^2	x	-2
x	x^3	x^2	$-2x$
3	$3x^2$	$3x$	-6

$(x^2 + x - 2)(x + 3) = x^3 + 3x^2 + x^2 + 3x - 2x - 6$

$(x - 1)(x + 2)(x + 3) = \mathbf{x^3 + 4x^2 + x - 6}$

Notice that the constant term in the expansion is the product (that means times) of the numerical part in the brackets. $1 \times -2 \times 3 = -6$. This is important for helping us to factorise cubic expressions later on, but can be a nice way to check whether your work is likely to be correct.

Your Turn:

Expand and simplify:

1. $5(2x - 7)$

2. $8x(2x + 3)$

3. $7a(3a + 2b - 4)$

4. $5(2x + 1) + 3(x + 4)$

5. $8y(y - 4) - 2y(3 - y)$

6. $(3x + 2)(x + 5)$

7. $(x - 4)(3x - 9)$

8. $(a + b)(b - c)$

9. $(3x + 2)^2$

10. $(x + 8)(2x + y - 4)$

11. $(x + 3)(x + 4)(x + 1)$

12. $(2x - 5)(x - 2)(x + 7)$

13. $(x + 1)^3$

14. $(x + 2)^2(x + 5)$

For answers, go to page 100.

Factorising

To *factorise* means to put an expression back into brackets. To do this, begin by taking out any common factors among the terms (that's where the word comes from!). Then divide each term by this number, or expression, to find the terms that go inside the brackets.

E.g. Factorise $8x^2 - 10x$

The common factor is $2x$, so $2x$ goes on the outside of the brackets. $8x^2 \div 2x = 4x$ and $10x \div 2x = 5$.

$8x^2 - 10x = \mathbf{2x(4x - 5)}$

A common misconception is to think that, because an expression contains more than two terms or has a squared term in it, it must factorise into double brackets. This is not always true.

E.g. Factorise $20p^2q + 5pq^2 - 15pq$

The common factor is $5pq$, so $5pq$ goes on the outside of the brackets.
$20p^2q \div 5pq = 4p$, $5pq^2 \div 5pq = q$ and $-15pq \div 5pq = -3$.

$20p^2q + 5pq^2 - 15pq = \mathbf{5pq(4p + q - 3)}$

You might even see an expression where the common factor is itself an expression.

E.g. Factorise $4(x + y) - p(x + y)$

The common factor is $(x + y)$. $4(x + y) \div (x + y) = 4$ and $-p(x + y) \div (x + y) = -p$.

$4(x + y) - p(x + y) = \mathbf{(x + y)(4 - p)}$

Your Turn:

Factorise fully:

1. $12x + 15$

2. $27x - 18$

3. $10y^2 + 28y$

4. $14ab + 21a$

5. $32x + 40y - 24$

6. $10x^2y - 15xy^2$

7. $12a^3b^2 + 18a^2b^3 - 27ab^4$

8. $a(b + c) + 5(b + c)$

9. $x(y + 3) + 2(y + 3)$

10. $2r(a - 4) - p(a - 4)$

For answers, go to page 101.

Factorising Quadratic Expressions

Quadratic expressions are of the form $ax^2 + bx + c$, where $a \neq 0$. a and b are called the coefficients of x^2 and x respectively. A quadratic is a polynomial, but since these appear so often in the AS and A-level course, they get their own section!

Factorising: When $a = 1$

When $a = 1$, the expression is $x^2 + bx + c$. If it can be factorised, this sort of expression will go into two brackets, with an x at the front of each. To find the numerical part, find two numbers that multiply to give c and add to give b.

E.g. Factorise $x^2 + 2x - 15$

Find two numbers that multiply to give -15 and add to give 2. List the factors of 15 then deal with the signs. The factors of 15 are 1 and 15, or 3 and 5.

A negative multiplied by a positive is a negative so one number in each factor pair will have to be negative. To give a sum of positive 2, we choose -3 and 5.

$x^2 + 2x - 15 = \mathbf{(x - 3)(x + 5)}$

Check your work by expanding the brackets and checking you get the original expression.

Your Turn:

Factorise fully:

1. $x^2 + 7x + 10$
2. $x^2 + 12x + 20$
3. $x^2 + 4x - 21$
4. $x^2 - x - 6$
5. $x^2 - 13x + 30$
6. $x^2 - 10x + 25$

For answers, go to page 103.

Factorising: The Difference of Two Squares

If the expression consists of two square numbers separated by a minus sign then it can be factorised using the difference of two squares rule.

$a^2 - b^2 = (a + b)(a - b)$

E.g. Factorise $x^2 - 25$

$5^2 = 25$, so this becomes $\mathbf{(x + 5)(x - 5)}$

This works even if the coefficient of x is not 1 (as long as it's square) or if both terms are algebraic.

E.g. Factorise $16x^2 - 49y^2$

$(4x)^2$ is $16x^2$ and $(7y)^2$ is $49y^2$, so this becomes $\mathbf{(4x + 7y)(4x - 7y)}$

Your Turn:

Factorise fully:

1. $x^2 - 36$
2. $a^2 - 81$
3. $4x^2 - 9$
4. $25a^2 - b^2$
5. $9x^2 - 100y^2$
6. $x^4 - y^2$

For answers, go to page 103.

Factorising – When $a \neq 1$

This is a little more involved and, luckily, you will have a calculator that can do most of this for you. However, you still need to be able to factorise this sort of expression.

There are a number of techniques: one of which is observation; another involves reversing the process for expanding brackets.

For an expression of the form $ax^2 + bx + c$, begin by multiplying a and c to get ac. Then, find a pair of numbers whose product is ac and whose sum is b. Next, split up the middle term into two x terms with these numbers as their coefficients. Finally, factorise each *pair* of terms.

E.g. Factorise $2x^2 + 9x + 10$

$2 \times 10 = 20$

Two numbers that have a product of 20 and a sum of 9 are 4 and 5.

$2x^2 + 9x + 10 = 2x^2 + 4x + 5x + 10$

Factorise the first pair of terms and the second.

$2x^2 + 4x + 5x + 10 = 2x(x + 2) + 5(x + 2)$

Finally, fully factorise by taking out a factor of $(x + 2)$

$2x(x + 2) + 5(x + 2) = \mathbf{(x + 2)(2x + 5)}$

Your Turn:

Factorise fully:

1. $2x^2 + 11x + 12$

3. $4x^2 + 8x - 21$

2. $3x^2 + 26x + 35$

4. $3x^2 - 19x + 20$

For answers, go to page 104.

Completing the Square

By completing the square, we can solve non-factorable quadratic equations, perform proofs and identify turning points on quadratic graphs. The completed square form for the expression $x^2 + bx + c$ is $(x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$. In other words, we halve the coefficient of x to find the numerical part inside the brackets. Then, square this and subtract it from the bracketed expression.

E.g. Write $x^2 - 10x + 3$ in the form $(x + m)^2 + n$, where m and n are integers.

Begin by halving the coefficient of x to find the value of m. Then, square this number and subtract it.

$-10 \div 2 = -5$

$x^2 - 10x + 3 = (x - 5)^2 - 5^2 + 3$

$= \mathbf{(x - 5)^2 - 22}$

When we have an expression in completed square form, we can find the turning point. A quadratic graph whose equation is $y = (x + m)^2 + n$ has a turning point at $(-m, n)$.

Take our previous example. The quadratic equation $y = x^2 - 10x + 3$ can be written as $y = (x - 5)^2 - 22$. This means that the quadratic graph has a turning point at (5, -22).

E.g. A curve is given by the equation $y = x^2 + 4x - 1$. Write the equation in the form $y = (x + m)^2 + n$. Hence, write down the coordinates of the turning point of this graph.

$4 \div 2 = 2$

$x^2 + 4x - 1 = (x + 2)^2 - 2^2 - 1$

$= \mathbf{(x + 2)^2 - 5}$

The turning point has coordinates (-2, -5).

Note that, if the coefficient of x^2 is not equal to 1, you will need to factorise part of the expression first.

E.g. $3x^2 + 6x + 5 = 3(x^2 + 2x) + 5$

$= 3((x + 1)^2 - 1^2) + 5$

$= 3(x + 1)^2 - 3 + 5$

$= \mathbf{3(x + 1)^2 + 2}$

Your Turn:

Write each equation in completed square form, and then find the coordinates of the turning point.

1. $y = x^2 + 8x + 23$

2. $y = x^2 - 6x + 1$

3. $y = x^2 + 4x - 6$

4. $y = x^2 + 3x + 9$

5. $y = x^2 - 5x - 8$

6. $y = 2x^2 + 12x + 7$

7. $y = 3x^2 + 12x + 2$

8. $y = 2x^2 + 6x + 23$

For answers, go to page 105.

Linear Equations and Inequalities

Solving Equations and Inequalities

To solve a linear equation or inequality, we apply a series of inverse operations to isolate the variable (usually the letter x). If there are any fractions or brackets in the equation, it is sensible to deal with those first, then find a way to collect all the letters on one side of the equation and all the numbers on the other.

Example 1

Solve $\frac{4x + 5}{3} = 2(x - 7)$

Begin by multiplying both sides by 3 to remove the denominator of the fraction.

$4x + 5 = 6(x - 7)$

Next, multiply out the brackets. Sometimes, you will be able to divide through by the number in front but here that doesn't help us!

$4x + 5 = 6x - 42$

Gather the letters on one side by subtracting $4x$.

$5 = 2x - 42$

Gather the numbers on the other side by adding 42.

$47 = 2x$

Divide through by 2. It can be sensible to leave the answer in a fraction in its simplest form.

$\mathbf{\frac{47}{2} = x}$

When working with inequalities, if we divide or multiply **both sides** by a negative number, we must remember to reverse the inequality symbol.

Example 2

Solve $-5 < 7 - 3x \leq 11$

Begin by subtracting 7 from all sides.

$-12 < -3x \leq 4$

Next, divide by -3 and reverse the inequality symbols.

$\mathbf{4 > x \geq -\frac{4}{3}}$

It is convention to write this answer as $\mathbf{-\frac{4}{3} \leq x < 4}$ with the lower number in the range appearing first.

Your Turn:

1. Solve the following equations:

   a. $8(2x + 3) = 24$

   b. $\frac{3x - 4}{2} = 5$

   c. $2(\frac{3(x + 1)}{5}) = 6$

   d. $4(2x - 5) = 3(x + 2)$

   e. $\frac{5x - 7}{x} = 9$

   f. $8 - \frac{3x}{2 + x} = 10$

2. Solve the following inequalities:

   a. $8x + 3 > 2(x + 5)$

   b. $\frac{2x - 1}{7} \leq 3$

   c. $7 \leq 4x + 5 < 19$

   d. $5(3 - 2x) \geq 1$

3. Find the set of solutions which satisfies the following inequalities:

   $8x \geq 5 - x$ and $-4 < 3x + 1 \leq 10$

For answers, go to page 106.

Graphs

Equations of Straight-Line Graphs

The general equation of a straight line is $y = mx + c$, where m represents the gradient (steepness) of the line and c is the value of the y-intercept. You might need to rearrange an equation to be able to calculate one of these.

Example 1

Find the gradient and the coordinates of the y-intercept of the line with equation $3x + 2y = 6$.

Begin by rearranging to make y the subject.

$2y = 6 - 3x$

$y = 3 - \frac{3}{2}x$

The gradient is $\mathbf{-\frac{3}{2}}$ and the y-intercept is 3, so its coordinates are **(0, 3)**.

You can find the gradient of a straight line given any two points on that line. If those two points are (x_1, y_1) and (x_2, y_2), the formula for the gradient of a straight line is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Parallel and Perpendicular Lines

Two lines are parallel if their gradients are equal, and they are perpendicular if the product of their gradients is -1. In other words, if m_1 and m_2 are the gradients of two lines, these lines are perpendicular if $m_1 \times m_2 = -1$.

Example 2

Prove that the line passing through the points (4, 1) and (2, 6) is perpendicular to the line whose equation is $5y = 2x + 3$.

The gradient of the first line is $m = \frac{6 - 1}{2 - 4} = -\frac{5}{2}$

The equation of the second line can be written as $y = \frac{2}{5}x + \frac{3}{5}$, so its gradient is $\frac{2}{5}$

$\mathbf{-\frac{5}{2} \times \frac{2}{5} = -1}$ **therefore the lines are perpendicular.**

Note that it's sensible to leave any non-integer values in fractional form. Fractions are *generally* easier to work with and if the equivalent decimal is non-terminating then they also provide an exact solution.

If you know the gradient of a line and the coordinates of at least one point, you can find the equation. Substitute everything you know into the equation of the line and then solve for c.

Example 3

Find the equation of the line which is parallel to the line with equation $y = 4x + 2$ and which passes through (3, 7).

The gradient of the line given is 4 and the lines are parallel, so the gradient of our line is also 4.

Substitute this value into the equation $y = mx + c$ to get $y = 4x + c$.

We know the line passes through (3, 7), so substitute $x = 3$ and $y = 7$ into this equation and solve for c.

$7 = 4 \times 3 + c$

$7 = 12 + c$

$c = 7 - 12 = -5$

The equation is $\boldsymbol{y = 4x - 5}$

Your Turn:

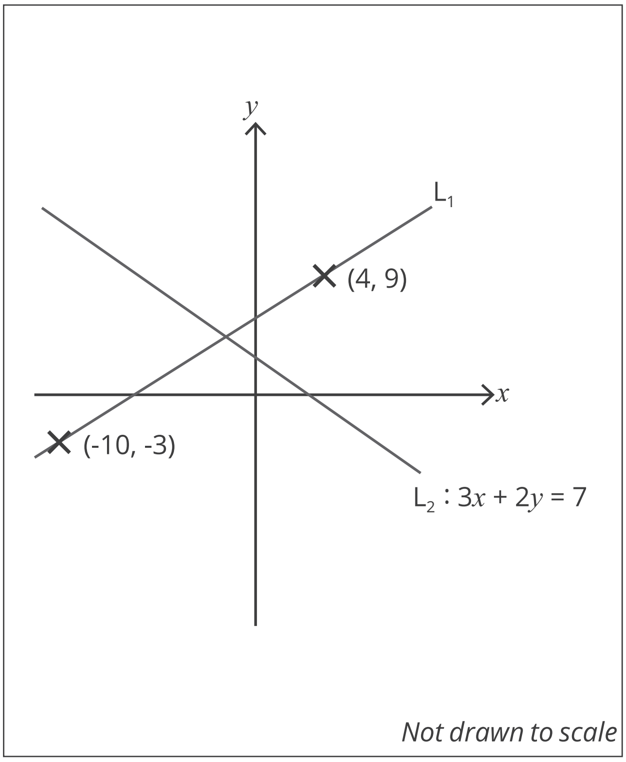
1. Complete the table:

Equation	Gradient	y-intercept
$y = 3x + 7$		
$y = 2 - x$		
$2y = x + 5$		
$3y + 2x = -1$		
	-2	0
	$\frac{3}{4}$	-1

2. Find the equation of the line passing through the points with coordinates (2, 3) and (4, 1).

3. A line whose gradient is $\frac{1}{3}$ passes through the point (-6, 9). Work out the equation of this line, giving your answer in the form $ay + bx = c$, where a, b and c are integers.

4. The diagram shows two straight lines. Are the lines perpendicular? Justify your answer.



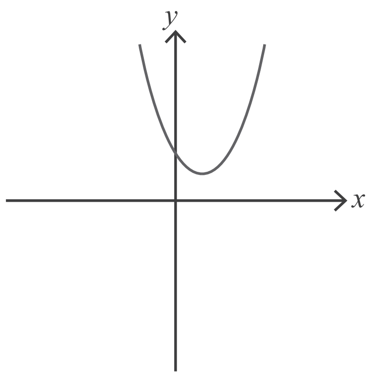
5. Does the line with equation $2x + 5y = -1$ pass through the point with coordinates (2, -1)?

For answers, go to page 108.

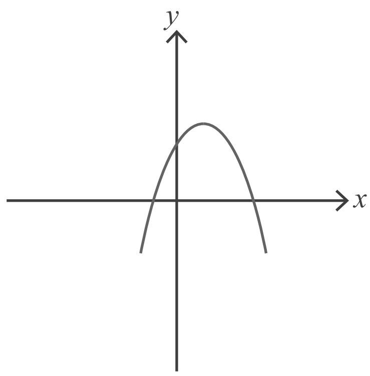
Quadratic Graphs

All quadratic graphs are parabolas: a symmetrical curve. The orientation of this parabola depends on whether the coefficient of x^2 is positive or negative.

If the coefficient of x^2 is positive, the graphs are u-shaped:



If the coefficient of x^2 is negative, the graphs are n-shaped:



We can use our skills for working with quadratic equations to find other key features of the graph:

1. The y-intercept is found by setting $x = 0$ and solving the given equation for y.
2. The x-intercept is found by setting $y = 0$ and solving the given equation for x.
3. The turning point (which will be the minimum or maximum of a quadratic function) can be found by completing the square. This can also be found by finding the average of the x-intercept values but completing the square can be more efficient. A graph with an equation of the form $y = a(x + b)^2 + c$ has a turning point at $(-b, c)$.

For example,

Sketch the graph of $y = x^2 - 7x + 10$, clearly indicating any points of intersection with the axes and the location of the turning point of the curve.

The coefficient of x^2 is 1. This is positive, so our graph will be u-shaped.

The y-intercept is found by substituting $x = 0$ into the equation.

$y = 0^2 - 7 \times 0 + 10$

$y = 10$

The x-intercept is found by substituting $y = 0$ into the equation then solving the resulting equation.

$0 = x^2 - 7x + 10$

$0 = (x - 2)(x - 5)$

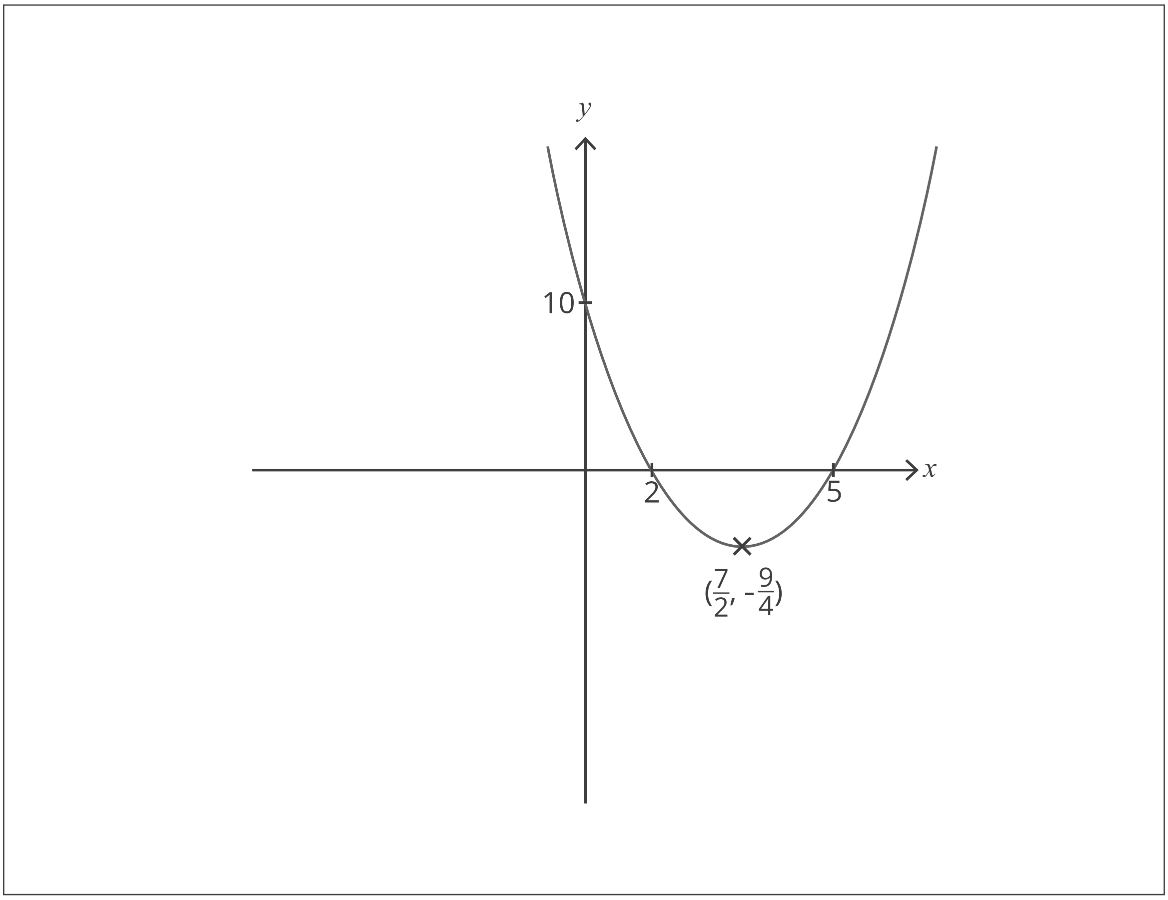
$x = 2, x = 5$

The turning point is found by completing the square.

$$\begin{aligned} x^2 - 7x + 10 &= (x - \tfrac{7}{2})^2 - (\tfrac{7}{2})^2 + 10 \\ &= (x - \tfrac{7}{2})^2 - \tfrac{9}{4} \end{aligned}$$

The turning point has coordinates $\mathbf{(\frac{7}{2}, -\frac{9}{4})}$.

When sketching a graph, it does not need to be to scale but should be the right shape and roughly in proportion as shown:



Your Turn:

1. Consider the curve with equation $y = x^2 + 4x - 5$.

   a. Find the coordinates of the point where this curve intersects the y-axis.

   b. Find the coordinates of the points where this curve intersects the x-axis.

   c. Hence, sketch the graph of $y = x^2 + 4x - 5$, clearly indicating any points of intersection with the axes.

2. Consider the curve with equation $y = x^2 + 8x - 1$.

   a. Find the coordinates of the turning point of this curve.

b. State whether the turning point is a maximum or minimum. Justify your answer.

3. Sketch the graph of $y = x^2 + 4x - 21$, clearly indicating any points of intersection with the axes and the location of the turning point of the curve.

4. Sketch the graph of $y = -x^2 + 7x$, clearly indicating any points of intersection with the axes.

5. Sketch the graph of $y = 2x^2 + 17x + 8$, clearly indicating any points of intersection with the axes.

For answers, go to page 109.

Quadratic Equations and Inequalities

To solve quadratic equations and inequalities, begin by checking if the quadratic part can be factorised. If it can, then this is the most efficient method to use.

Example 1

Solve $2x^2 + 9x + 7 = 0$

Start by factorising the quadratic part to get:

$(2x + 7)(x + 1) = 0$

For this equation to be correct, either of the expressions in the brackets must be equal to zero.

$2x + 7 = 0$ or $x + 1 = 0$

Solve each linear equation to get $\boldsymbol{x = -\frac{7}{2}}$ or $\boldsymbol{x = -1}$.

We can always check the solutions by substituting them back into the original equation and making sure the result is zero.

We need to be a little bit careful when solving quadratic inequalities. We begin by using the same method: Set the quadratic part equal to zero and solve. This tells us where the graph crosses the x-axis. By sketching the quadratic curve, we can then decide which set of x-values makes the inequality true.

Example 2

Solve $x^2 + x - 20 > 0$

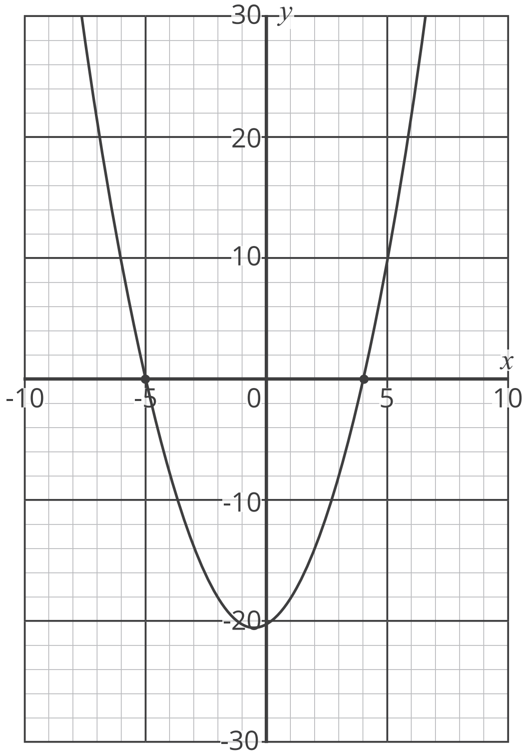
Set equal to zero and solve:

$x^2 + x - 20 = 0$

$(x + 5)(x - 4) = 0$

$x = -5$ or $x = 4$

The graph of the equation $y = x^2 + x - 20$ looks like this:



The function is greater than zero for the parts of the curve that lie above the x-axis. That is when $\boldsymbol{x < -5}$ and $\boldsymbol{x > 4}$.

We can also complete the square to help us find exact solutions to quadratic equations that cannot be factorised.

Example 3

Solve $x^2 - 6x + 1 = 0$

Begin by completing the square for the quadratic part.

$(x - 3)^2 - 10 = 0$

Next, solve by performing a series of inverse operations.

$(x - 3)^2 = 10$

When we take the square root, we must find both the positive and negative square root of the numerical part.

$x - 3 = \pm\sqrt{10}$

$x = 3 \pm \sqrt{10}$

The two solutions are $\boldsymbol{x = 3 + \sqrt{10}}$ and $\boldsymbol{x = 3 - \sqrt{10}}$

Your Turn:

1. Solve by factorising:

   a. $x^2 - 11x + 28 = 0$

   b. $3x^2 - 16x - 12 = 0$

   c. $y^2 + 4y - 35 = 2y$

   d. $6n^2 - 6n = -n^2$

2. Solve the inequalities by sketching the graph:

   a. $x^2 + 12x + 32 \geq 0$

   b. $x^2 - 2x - 8 < -x - 2$

c. $4x^2 + 20x - 11 \leq 0$

d. $8x^2 + 9x > 2x$

3. Solve by completing the square, writing surds in their simplest form:

   a. $x^2 + 8x + 3 = 0$

   b. $x^2 + 3x - 7 = 0$

   c. $2x^2 + 12x + 1 = 0$

   d. $x^2 + 16x + 3 = -9$

For answers, go to page 112.

The Quadratic Formula

For quadratic equations that cannot be factorised, the quadratic formula can help. For an equation of the form $ax^2 + bx + c = 0$, the solutions are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that, since the equation includes a square root, there won't always be real solutions to a quadratic equation. The part inside the square root sign is called the discriminant and is referred to as the delta symbol Δ. It follows that if this is less than zero then there are no real solutions to the equation. If it is equal to zero, there is one solution, and if it is greater than zero, there are two.

Example 1

Solve the equation $3x^2 + 5x - 4 = 0$, giving your answers correct to 3 significant figures.

$a = 3$, $b = 5$ and $c = -4$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times -4}}{2 \times 3}$$

x = -2.26 or **x = 0.591**

Your Turn:

1. Solve $2x^2 + 5x + 1 = 0$, giving your answers correct to 3 significant figures.

2. Solve $5x^2 + 2x = 19$, giving your answers correct to 3 significant figures.

3. Explain why the graph of the equation $y = x^2 + 4x + 9$ does not intersect the x-axis.
   Use the discriminant $\Delta = b^2 - 4ac$

For answers, go to page 115.

Simultaneous Equations

When solving simultaneous equations, we are finding the values of the given variables (usually x and y) that make both equations true. There are two techniques: elimination (this works better for linear equations) and substitution (this works better for quadratic equations but can be used in either case).

Example 1

Solve the simultaneous equations $2x - 4y = -14$ and $3x + 7y = 18$.

To use the elimination method, we need to multiply one or both equations by some constant to create a common coefficient.

Let's call $2x - 4y = -14$ equation 1 and $3x + 7y = 18$ equation 2.

Multiply equation 1 by 3 and equation 2 by 2.

$6x - 12y = -42$

$6x + 14y = 36$

To eliminate the x parts, subtract one equation from the other (if the signs of the coefficients are different then we add the equations) and then solve the resulting equation. It doesn't matter which way around we do this, as long as we are careful with the signs.

$-26y = -78$

$y = 3$

Now, substitute this value into either of the original equations and solve for x. Take equation 1:

$2x - 4 \times 3 = -14$

$2x - 12 = -14$

$2x = -2$

$x = -1$

The solution is $\boldsymbol{x = -1}$ and $\boldsymbol{y = 3}$.

Example 2

Solve the simultaneous equations $x^2 + y^2 = 5$ and $y - x = 3$.

To use the substitution method, rearrange the linear equation to make y the subject.

$y = 3 + x$

Now, substitute this into the quadratic equation then solve.

$x^2 + (3 + x)^2 = 5$

$x^2 + 9 + 6x + x^2 = 5$

$2x^2 + 6x + 9 = 5$

$2x^2 + 6x + 4 = 0$

We can factorise from here but it is easier to divide through by 2, then factorise.

$x^2 + 3x + 2 = 0$

$(x + 2)(x + 1) = 0$

$x = -2$, $x = -1$

Don't forget to substitute both of these back into the original linear equation to find the corresponding values for y.

When $x = -2$:	When $x = -1$:
$y - (-2) = 3$	$y - (-1) = 3$
$y + 2 = 3$	$y + 1 = 3$
$y = 1$	$y = 2$

The solutions are $\boldsymbol{x = -2}$, $\boldsymbol{y = 1}$ and $\boldsymbol{x = -1}$, $\boldsymbol{y = 2}$.

Your Turn:

1. Solve each pair of simultaneous equations:

a. $x + y = 14$

$2x - y = 16$

c. $2x + 5y = 26$

$y = x + 1$

b. $3x + 2y = -4$

$2x + y = -3$

d. $x^2 + y^2 = 10$

$y = x + 2$

e. $2x^2 = y^2 - 8$
$y - x = 2$

For answers, go to page 116.

Geometry

A level will expand on what you've already learnt about vectors, with new ways to express them (including in 3 dimensions) and how to apply them to real-life situations. This will be particularly important when you start studying Mechanics, which uses vectors to model how forces and objects interact in the real world.

On the other hand, how you use trigonometry might start to seem less tied to the geometry of real-world objects. Up to now, you have mostly worked with trigonometry in terms of triangles, although what you've learnt about trigonometric graphs will have hinted that the trigonometric functions do not need to be directly related to shape. At A level, you will learn how to use trigonometric functions in more abstract problems. This is knowledge that can later be applied to more complex geometries, particularly those involving waves: sound waves in acoustics, electrical currents in electronic devices, electromagnetic waves in communications and more.

Right-angled Trigonometry

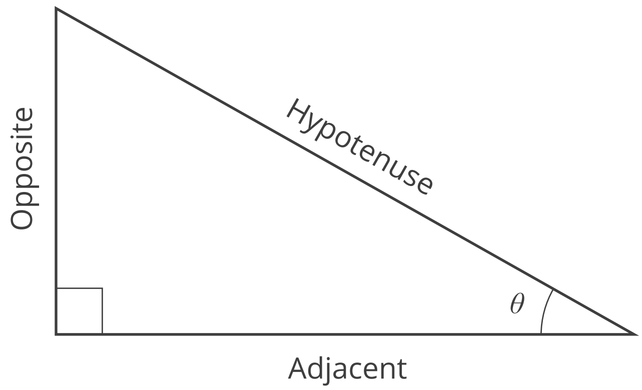
Right-angled trigonometry allows you to find a missing angle or side in a right-angled triangle when given two sides, or an angle and a side. Trigonometry uses the three trigonometric functions: sine, cosine and tangent, each of which can be expressed as a ratio of the length of two sides:

$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

Here, θ is the measure of the angle, O is the length of the opposite side, A is the length of the adjacent side and H is the length of hypotenuse.

The hypotenuse is always the side opposite the right angle and is always the longest side (this can be a good tool to check your answers make sense).

The opposite and adjacent sides are not fixed; they depend on the angle you are interested in. The opposite side is farthest from the angle (but not the hypotenuse). The adjacent side is next to the angle (but, again, not the hypotenuse).



Mnemonics

One of the challenges of solving a trigonometric problem is remembering the ratios. Some people use a mnemonic such as *SOH-CAH-TOA*. When written in the following format, this also lets you easily rearrange the formulae:

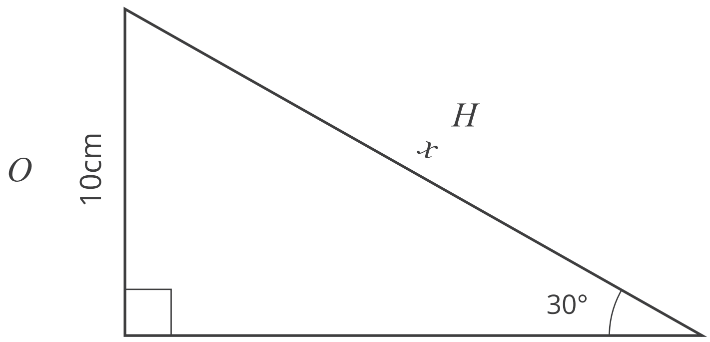
$$\frac{O}{S \times H} \qquad \frac{A}{C \times H} \qquad \frac{O}{T \times A}$$

Finding a Missing Side

Your first step in solving a right-angled trigonometric problem will be to label the sides of the triangle.

Your second step will be to choose the relevant ratio. In a typical problem, you will have a missing value and at least two known values. Consider the example below:

Find the length of side x.



Start by labelling the sides. In this case, we have the opposite (O) and hypotenuse (H). We know the angle and the opposite side but we are interested in the hypotenuse. Therefore, we need to choose the trigonometric ratio that includes O and H:

$$\sin\theta = \frac{O}{H}$$

We are looking for H, so we must rearrange our formula:

$$H = \frac{O}{\sin\theta}$$

Then, substitute our values and calculate our answer:

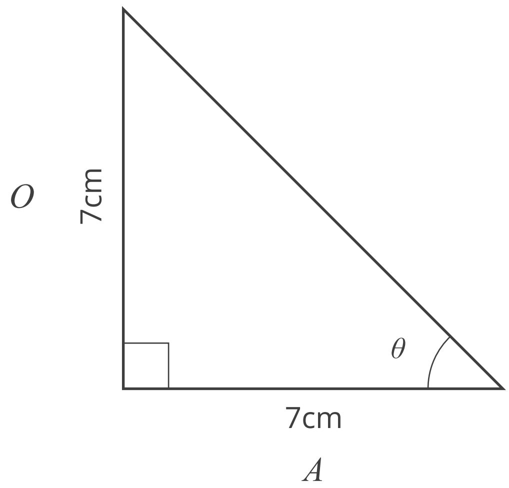
$$x = \frac{10}{\sin(30°)}$$

$$x = 20\text{cm}$$

We could also choose to rearrange the formula after substituting the values. This is entirely up to you, and won't affect the answer.

Finding a Missing Angle

In a similar way, you can find a missing angle when given two sides:



As before, start by labelling the sides. In this case, we have the opposite (O) and adjacent (A) sides. We need to use these to find the value of θ. We need to choose the trigonometric ratio that includes O and A:

$$\tan\theta = \frac{O}{A}$$

As before, we substitute our values:

$$\tan\theta = \frac{7}{7}$$

$$\tan\theta = 1$$

We have one more step. We are interested in the size of the angle θ but we still only know $\tan\theta$. To find θ, we need to carry out the inverse tan function. This is normally written as $\tan^{-1}$ or, sometimes, as arctan:

$$\tan\theta = 1$$

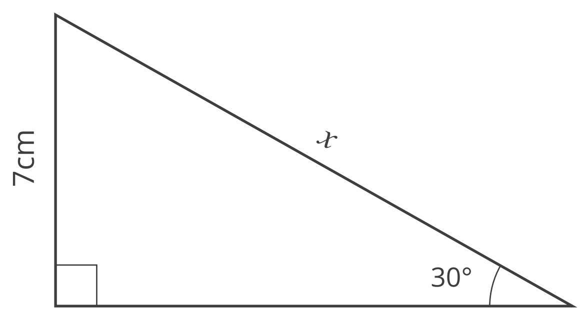
$$\theta = \tan^{-1}(1)$$

$$\theta = 45°$$

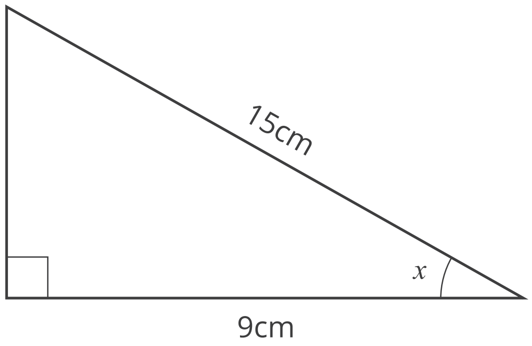
Your Turn:

1. In each question, find the value of x. Give your answers to 3s.f.

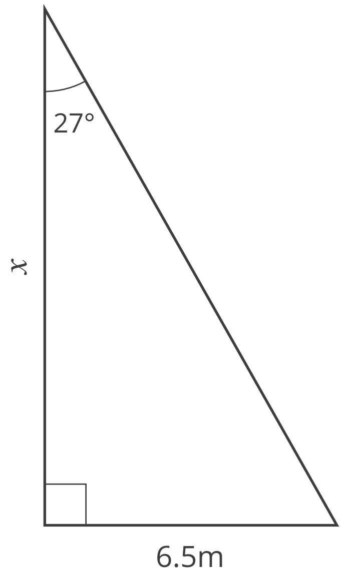
a.



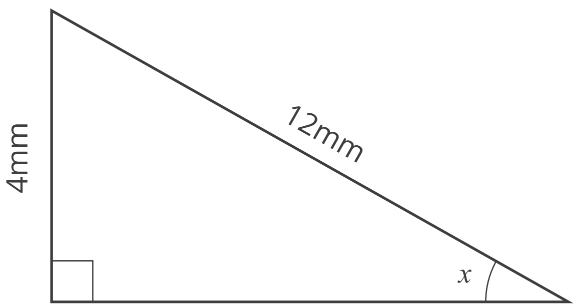
b.



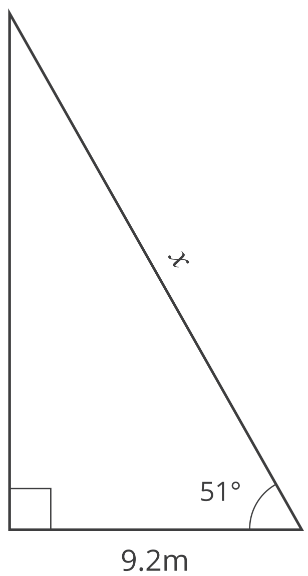
c.



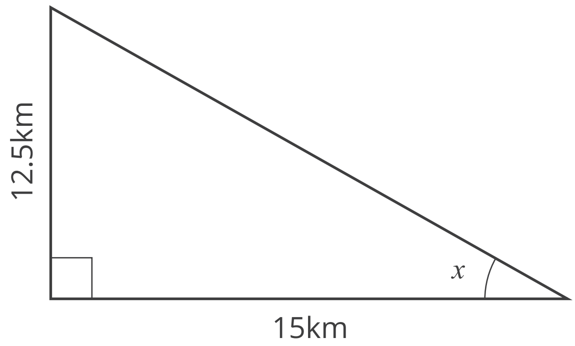
d.



e.



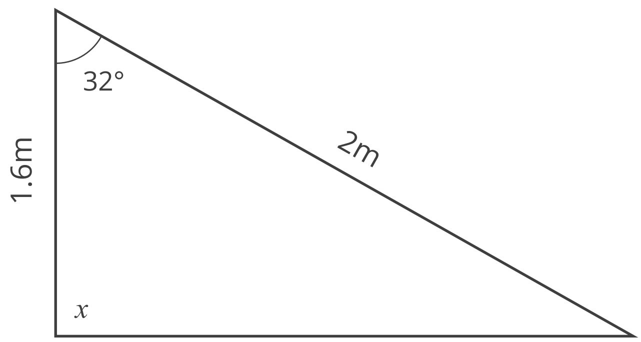
f.



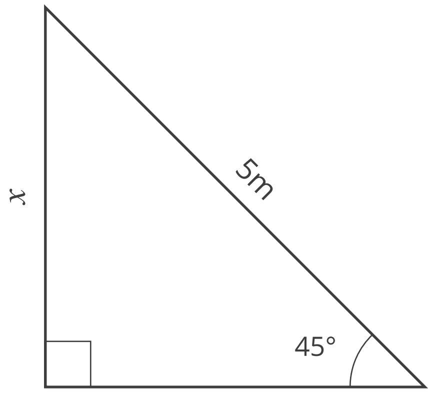
2. Zara is an engineer. She is building a wall, which needs to be supported as it is built. The ground is perfectly horizontal, the wall is perfectly vertical and the support is a straight steel beam. For the wall to be safe, the angle between the beam and the ground must be less than 55°.

   If the support beam is 12m long and the end of the support is 3.5m away from the base of the wall, is the wall safely supported?

3. Look at the diagram below. Is x a right angle? You must justify your answer.



4. Find the length of side x, giving your answer as a surd in its simplest form.



5. A mnemonic that can help to remember the trigonometric ratios is: **S**ydney **o**pera **h**ouse **c**an **a**lways **h**old **t**housands **o**f **A**ustralians. Try to make up your own mnemonic.

For answers, go to page 118.

Trigonometric Graphs

Graphs of Trigonometric Functions

In right-angled trigonometry problems, the values that your calculator provides have been limited by the question: an angle in a triangle must be greater than 0° and less than 180°, while an angle in a right-angled triangle must be less than 90°.

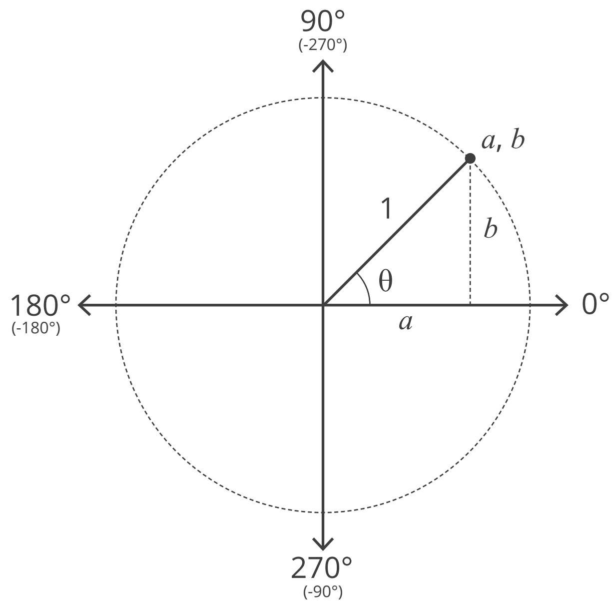
However, the scope of trigonometry stretches far beyond triangles.

Consider the function $f(x) = \sin x$. It has no limit on its input (x can be any number), and it will always give a single output, between $-1 \leq \sin x \leq 1$.

The inverse sine function, $f(y) = \sin^{-1} y$, is different. The input is limited: $-1 \leq y \leq 1$. If y is not between these values then $\sin^{-1} y$ is undefined. However, for a single input, there are an infinite number of possible outputs. You will need to be able to choose the output that best fits the problem you are working with. This is especially true when applying the sine and cosine rules, as these can apply trigonometric functions to triangles with angles that exceed 90°.

To better understand this, you need an understanding of what the trigonometric functions look like when graphed. You will need to be able to reproduce these graphs for any possible values but, luckily, they are periodic (they repeat the same pattern forever).

This can be shown using a unit circle. This is a circle with centre (0, 0) and radius 1. Plotting a point, (a, b), on the circumference of the circle gives us a right-angled triangle with a width of a and a height of b. The hypotenuse is equal to the radius, so is always 1.



If we consider the angle, θ, we can see that the value of $\cos\theta$ is equal to the adjacent divided by the hypotenuse. Since our hypotenuse is 1, this gives $\cos\theta = a$. Similarly, the value of $\sin\theta$ is equal to the opposite divided by the hypotenuse. Since our hypotenuse is 1, this gives $\sin\theta = b$.

We can see that both a and b will be between -1 and 1; this is simply because they are points on a circle radius 1.

We can plot the values of a or b against the size of the angle, θ, to draw graphs of the sine and cosine functions. Putting this information in a table gives:

θ / Function	0°	0° < θ < 90°	90°	90° < θ < 180°	180°	180° < θ < 270°	270°	270° < θ < 360°	360°
$b = \sin\theta$	0	Positive, increasing	1	Positive, decreasing	0	Negative, decreasing	-1	Negative, increasing	0
$a = \cos\theta$	1	Positive, decreasing	0	Negative, decreasing	-1	Negative, increasing	0	Positive, increasing	1

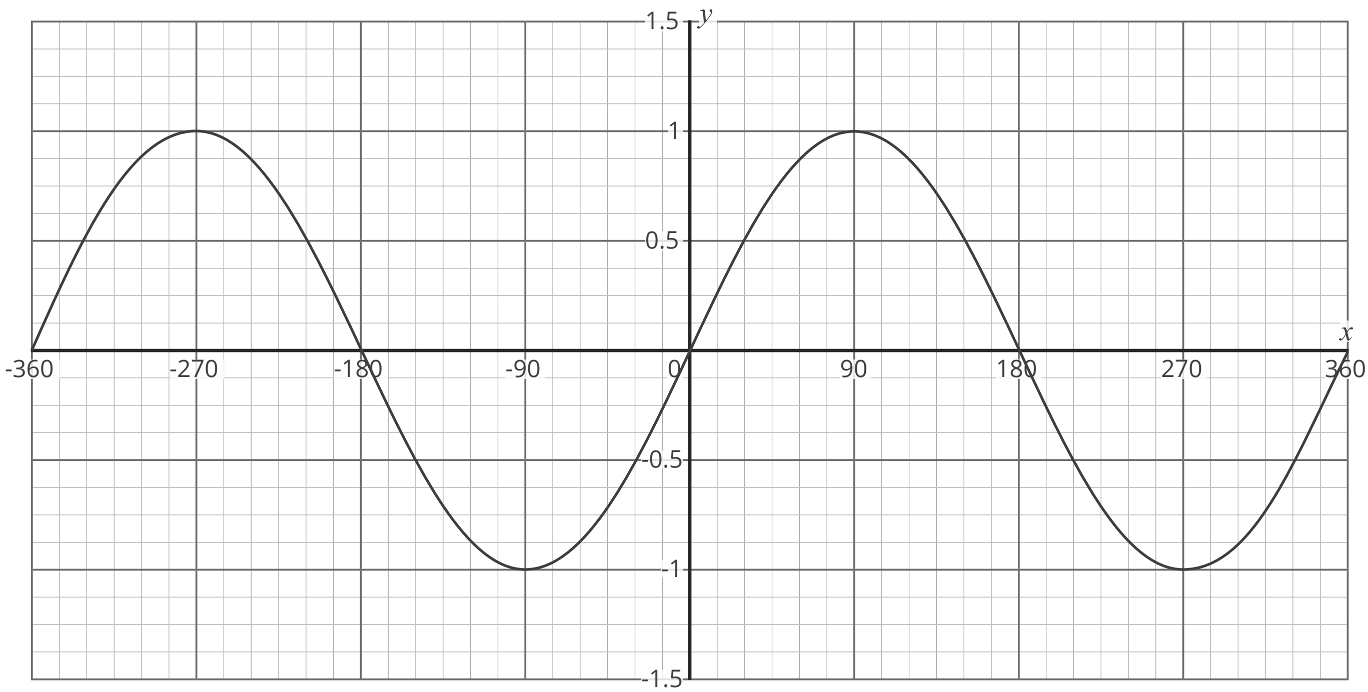
From the table, we can see that the values of a and b when θ = 0° are equal to the values of a and b when θ = 360°. This is because the functions and graphs are cyclical, every 360°.

Next, let's consider the value of $\tan\theta$. This isn't as simple as $\sin\theta$ and $\cos\theta$; $\tan\theta$ is opposite divided by adjacent. In this case, $\tan\theta = \frac{b}{a}$. When the angle, θ, is 90° or 270°, the length of the adjacent side, a, is 0. The result of dividing by 0 is undefined, giving us the following values for $\tan\theta$:

θ / Function	0°	0° < θ < 90°	90°	90° < θ < 180°	180°	180° < θ < 270°	270°	270° < θ < 360°	360°
$\tan\theta = \frac{b}{a}$	$\frac{0}{1} = 0$	Positive, increasing	$\frac{1}{0}$ (undefined)	Negative, increasing	$\frac{0}{-1} = 0$	Positive, increasing	$\frac{-1}{0}$ (undefined)	Negative, increasing	$\frac{0}{1} = 0$

We can plot the graphs of these functions:

$y = \sin x$

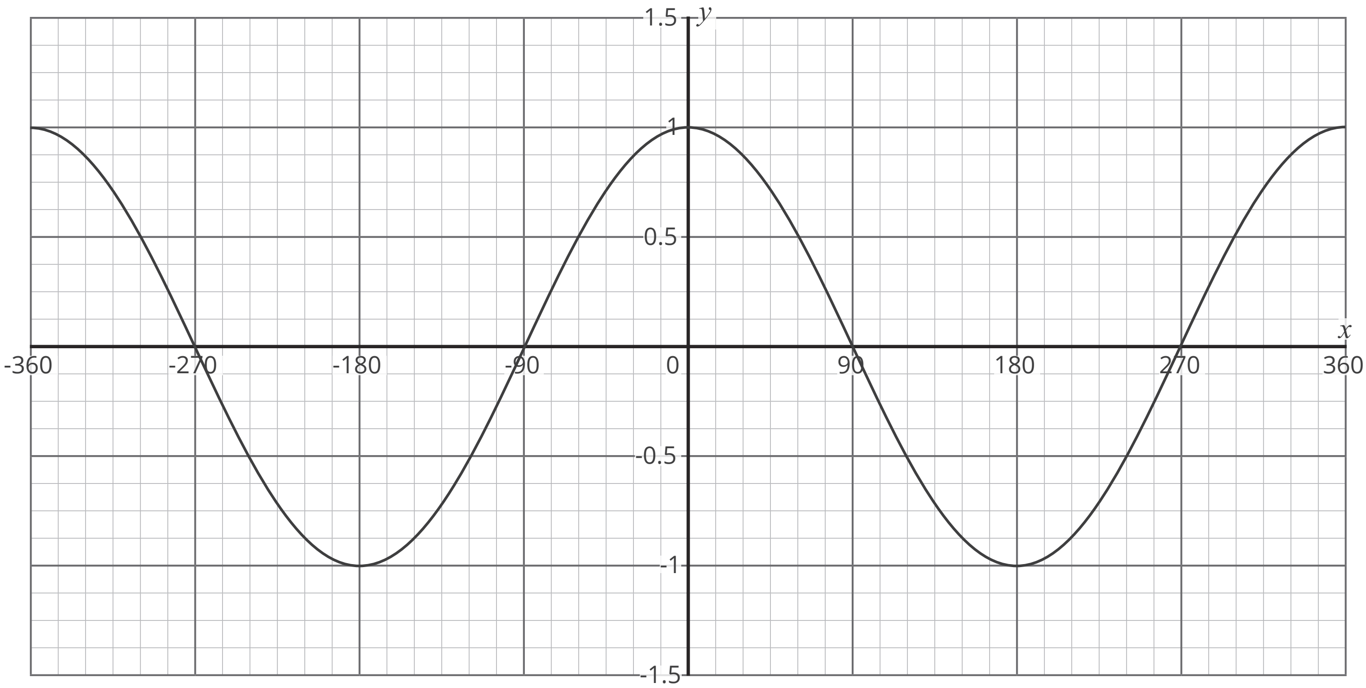


We can see the graph repeating every 360°; if we continued our axes on indefinitely then the curve will also repeat again and again. We can also see that each value on the x-axis is only associated with one value on the y-axis, for example sin(30°) = 0.5.

However, $\sin^{-1}y$ can be associated with a number of possible angles. For example, $\sin^{-1}(0.5)$ = -330°, -210°, 30° and 150°. As this graph repeats every 360°, $\sin^{-1}y$ is associated with an infinite number of possible angles.

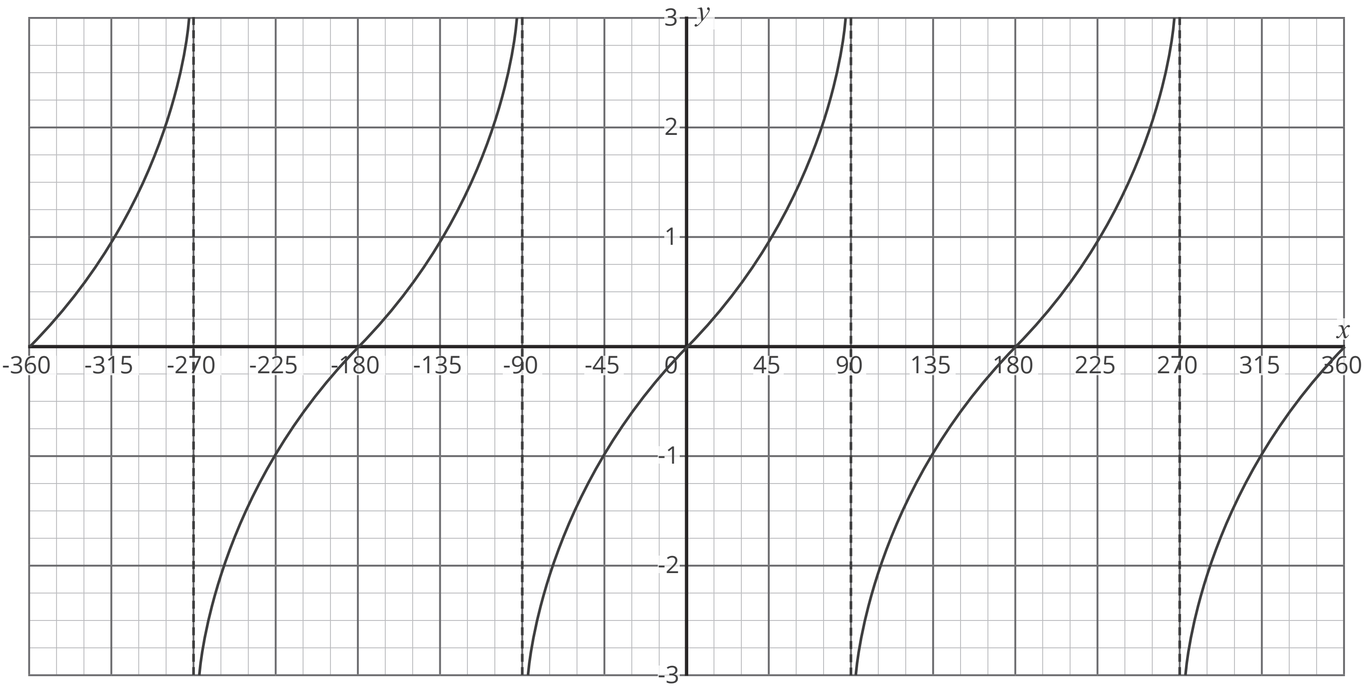
Therefore, you need to be able to choose the value that fits the problem that you are working with.

y = cos$x$



The graph of cos$x$ is similar to that of sin$x$. It repeats every 360° and the value of y is limited: $-1 \le \cos x \le 1$

y = tan$x$



The tan graph looks a little different. It is still periodic, but repeats every 180°, and its range is $\infty < \tan x < \infty$. As x approaches 90°, tan$x$ approaches infinity. The exact value of tan(90°) is undefined.

Example

x is an obtuse angle such that sin$x$ = 0.2. What is x, to 1d.p.?

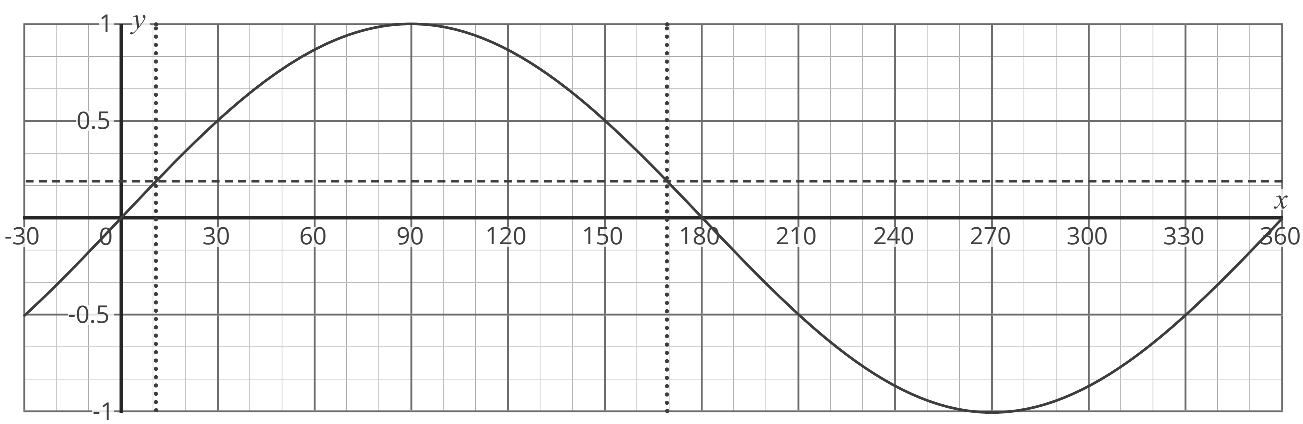
When you use the **arcsin** or **sin^{-1}** function on your calculator, it will give you an angle between 0° and 90°.

$\sin^{-1}(0.2) = 11.536...°$

This is convenient for right-angled trigonometry because, in a right-angled triangle, each non-right angle must be less than 90°.

But, in this question, x is obtuse, so 90° < x < 180°.

We need to look at the graph of y = sin$x$



We can use the symmetry of the graph about 90° to calculate the actual value of x.

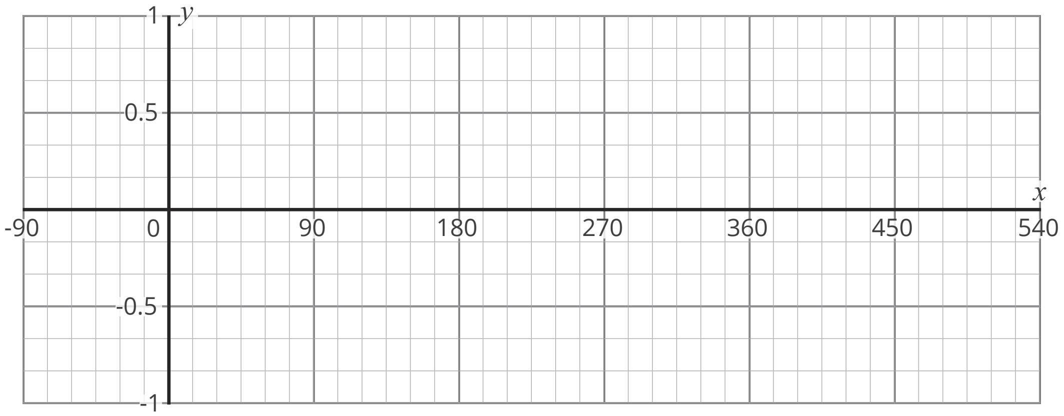
180 – 11.5 = **168.5°**

Your Turn:

1. x is an obtuse angle such that $\sin x = 0.7$. What is x, to 1d.p.?

2. x is a reflex angle such that $\cos x = -0.3$. What is x, to 3s.f.?

3. Sketch the graph of $y = \sin x$, where $360° \leq x \leq 540°$.



4. For each set of values of x, how many solutions are there for $\sin x = 0.2$?
   a. $0° \leq x \leq 360°$

   ____

   b. $-360° \leq x \leq 360°$

   ____

   c. $-180° \leq x \leq 180°$

   ____

5. For each set of values of x, how many solutions are there for $\cos x = -0.7$?
   a. $0° \leq x \leq 720°$

   ____

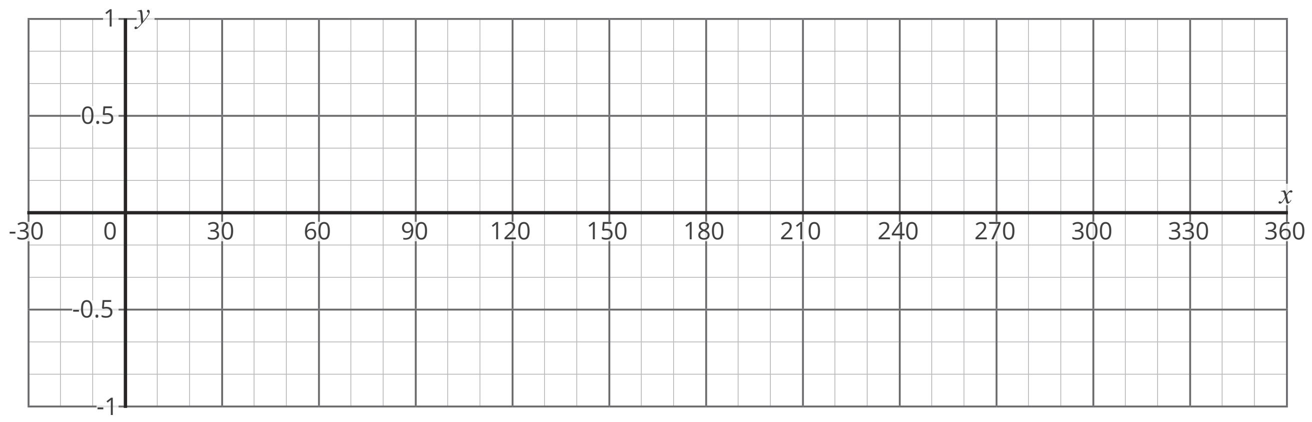
   b. $-360° \leq x \leq 0°$

   ____

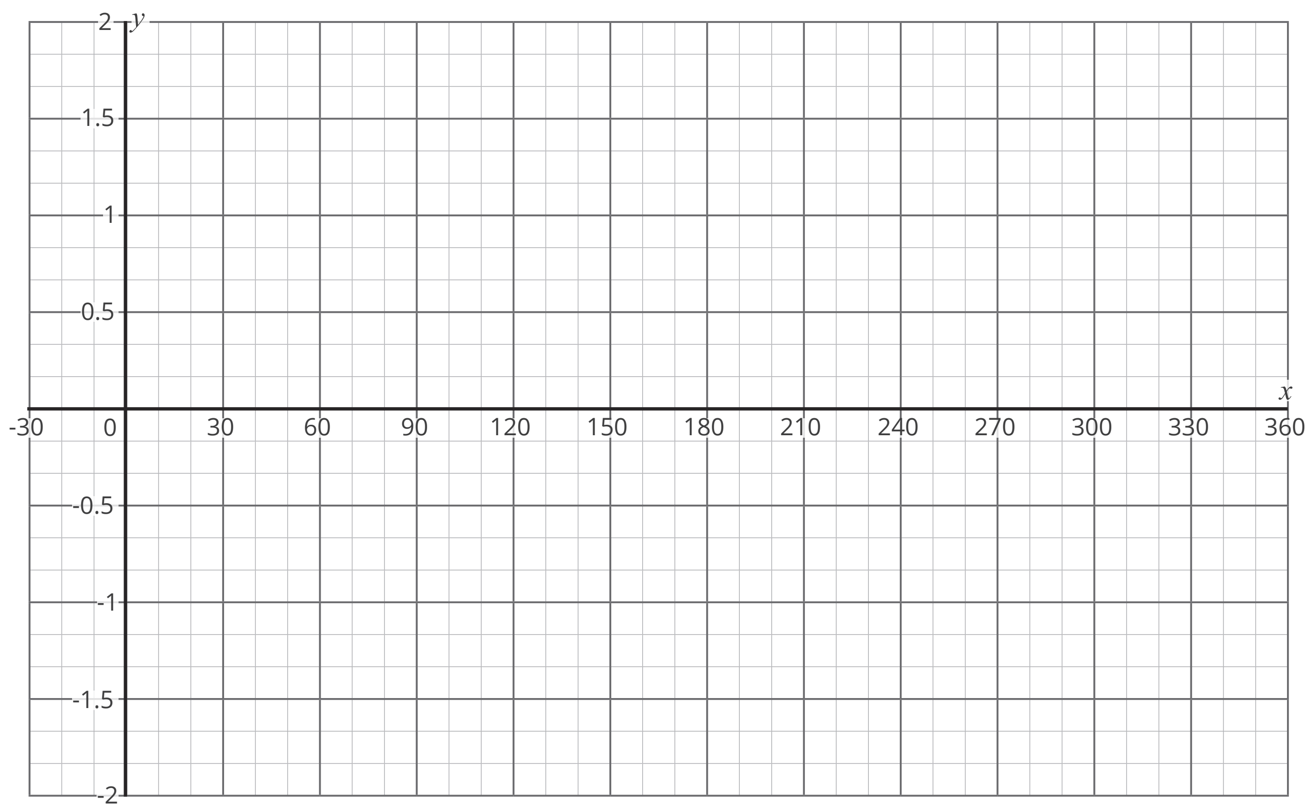
   c. $0° \leq x \leq 180°$

   ____

6. On the same set of axes, plot $y = \sin x$ and $y = \cos x$ for $0° \leq x \leq 360°$. Use your graph to find the values of x for which $\cos x = \sin x$



7. Plot $y = \tan x$ and $y = \cos x$ on the same set of axes, for $0° \leq x \leq 360°$. Write down the x value for each of the asymptotes for $y = \tan x$. What is the value of $y = \cos x$ for each of these values?



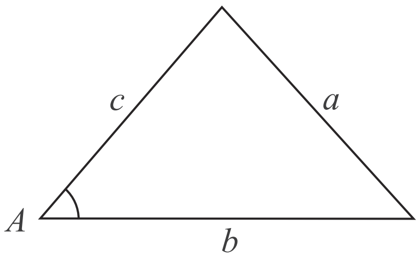
For answers, go to page 121.

Sine and Cosine Rules

The sine and cosine rules use trigonometric functions to find the size of missing angles or sides in any triangle. Unlike right-angled trigonometry, you do not need to have a right-angled triangle to apply them.

The Cosine Rule

For any triangle:



You may notice the similarity to Pythagoras' theorem. The cosine rule essentially uses the -2bccos$A$ term to apply Pythagoras' theorem to non-right-angled triangles. In a right-angled triangle, A is 90°, cos(90°) is 0 and so -2bccos$A$ is zero, leaving $a^2 = b^2 + c^2$.

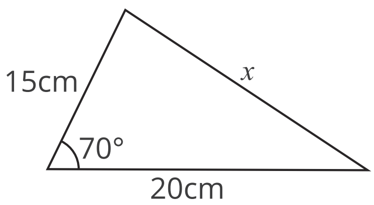
The cosine rule is

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

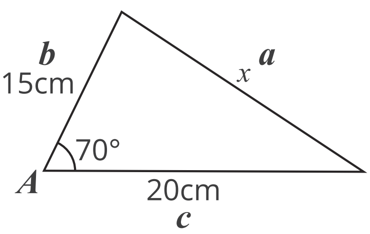
When using the cosine rule to find a missing side, you need the length of both adjacent sides and the angle opposite the side you are interested in.

Example 1

Find the length of side x. Give your answer correct to 3s.f.



Your first step is to label the sides and angles. Label the side you are interested in, a, and the angle opposite as A. Label the other two sides b and c.



Substitute your values into the formula and solve for x:

$a^2 = b^2 + c^2 - 2bc\cos A$

$x^2 = 15^2 + 20^2 - 2 \times 15 \times 20 \times \cos(70°)$

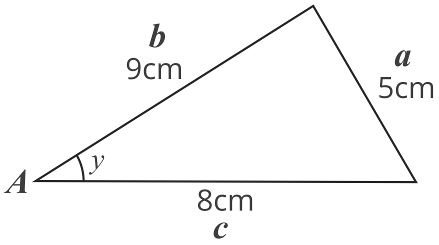
$x^2 = 419.787...$

$x = 20.5$cm

You can also use the cosine rule to find the size of an angle by rearranging the formula (or you may choose to learn the rearranged version of the formula by heart). To do this, you need to know the length of the side opposite the angle.

Example 2

Find the size of angle y. Give your answer correct to 1d.p.



In this case, the sides are already labelled. Remember: side a must be opposite angle A but sides b and c can be either way round. We now need to rearrange our formula:

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 - b^2 - c^2 = -2bc\cos A$ — Subtract b^2 and c^2 from both sides.

$\cos A = \dfrac{a^2 - b^2 - c^2}{-2bc}$ — Divide both sides by $-2bc$.

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ — Multiply top and bottom by $\frac{-1}{-1}$ to tidy up the negatives.

Now, you can substitute your values and solve for y:

$\cos y = \dfrac{9^2 + 8^2 - 5^2}{2 \times 9 \times 8}$

$\cos y = \dfrac{120}{144}$

$y = \cos^{-1}(\frac{120}{144})$

y = 33.6° (to 1d.p.)

Sine Rule

In problems where you don't have enough information to use the cosine rule, you may be able to use the sine rule instead. The sine rule says that within a triangle, the ratio of the length of each side to the sine of the opposite angle is equal:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

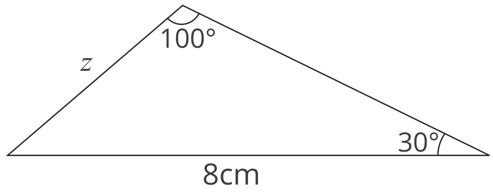
Or, if you take the reciprocal of each fraction:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

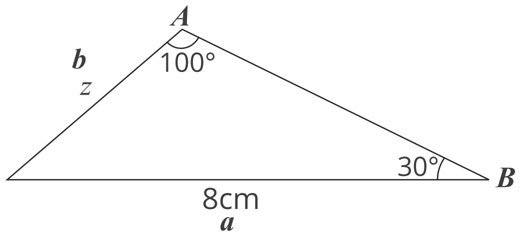
Where A is the angle opposite side a, B is the angle opposite side b and C is the angle opposite side c. We can then choose the sides we need for a question.

Example 3

Find the length of side z. Give your answer correct to 2d.p.



Start by labelling your sides:



We haven't labelled angle C or side c in this question as we don't need to use them. In the same way, we will remove them from the formula, leaving:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Substitute your values and solve for z:

$$\frac{8}{\sin(100°)} = \frac{z}{\sin(30°)}$$

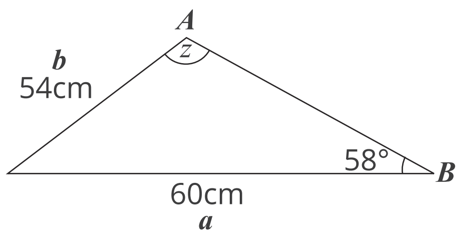
$$z = \frac{8 \times \sin(30°)}{\sin(100°)}$$

z = 4.06cm (to 2d.p.)

We can also use the sine rule to find a missing angle:

Example 4

Find the size of angle z. Give your answer correct to 3.s.f.



The sides are already labelled. This time, we are looking for an angle, so we use the formula:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Substitute your values and solve as before:

$$\frac{\sin z}{60} = \frac{\sin(58°)}{54}$$

$$\sin z = \frac{6 \times \sin(58°)}{54}$$

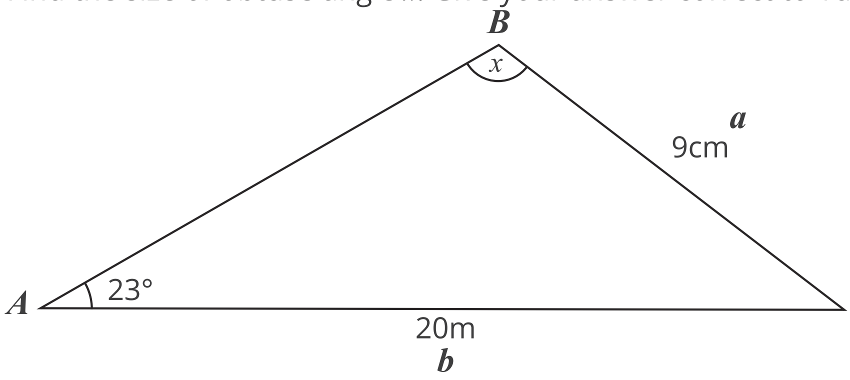
$$z = \sin^{-1}(0.942...)$$

z = 70.4° (to 3s.f.)

There are some situations in which you will need to be more careful when deciding how to apply the sine rule. Consider the following example:

Example 5

Find the size of obtuse angle x. Give your answer correct to 1d.p.



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin(23°)}{9} = \frac{\sin x}{20}$$

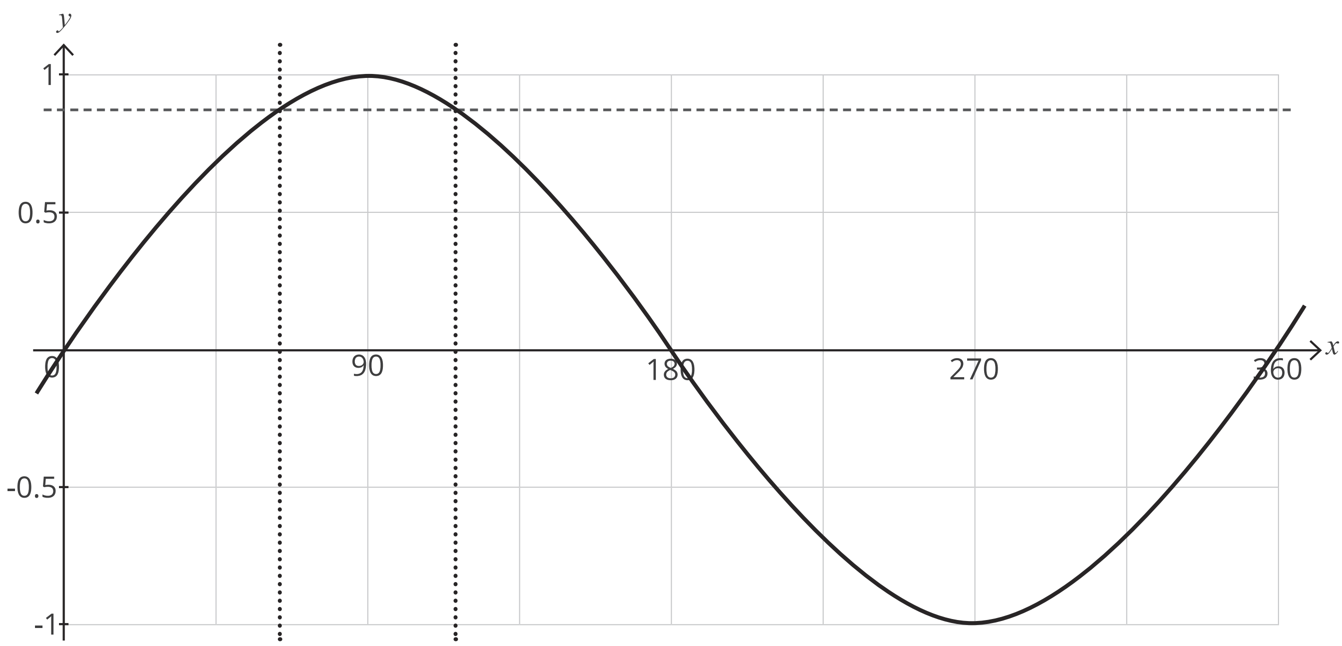
$$\sin x = \frac{20 \times \sin(23°)}{9}$$

$$x = \sin^{-1}(0.868...)$$

$$x = 60.260...°$$

At this point, you have to be careful. In non-right-angled triangles, the size of an angle can exceed 90°. While our calculations are correct, the answer is an acute angle, and the question states that angle x is obtuse.

Consider the graph of $y = \sin x$:



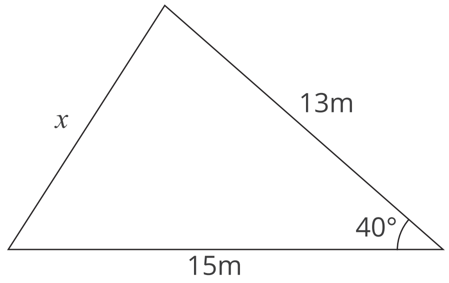
An angle in a triangle must be less than 180°. Within this range, there are two possible values for $\sin^{-1}(0.868)$ and we want to make sure we choose the one that corresponds to our angle. **This will not always be the answer given by your calculator**. In this case:

$x = 180 - 60.260...$

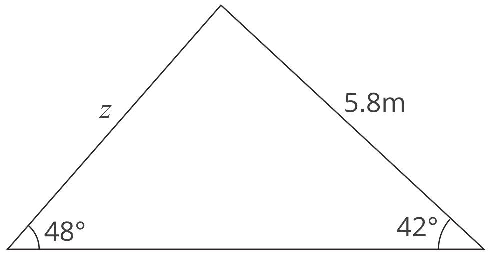
$\boldsymbol{x = 119.7°}$

Your Turn:

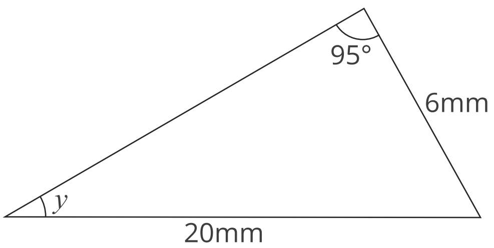
1. Find the length of side x, giving your answer correct to 1d.p.



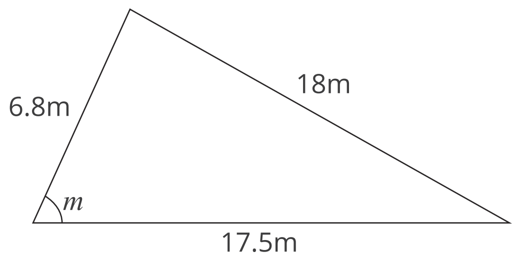
3. Find the length of side z, giving your answer correct to 1d.p.



2. Find the size of angle y, giving your answer correct to 3s.f.

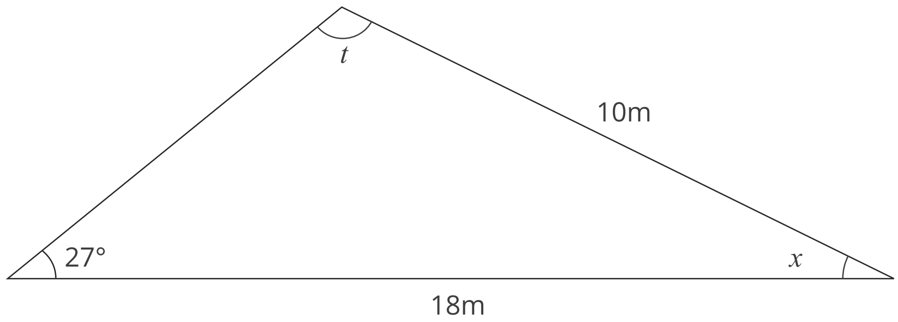


4. Find the size of angle m. Give your answer correct to 3s.f.

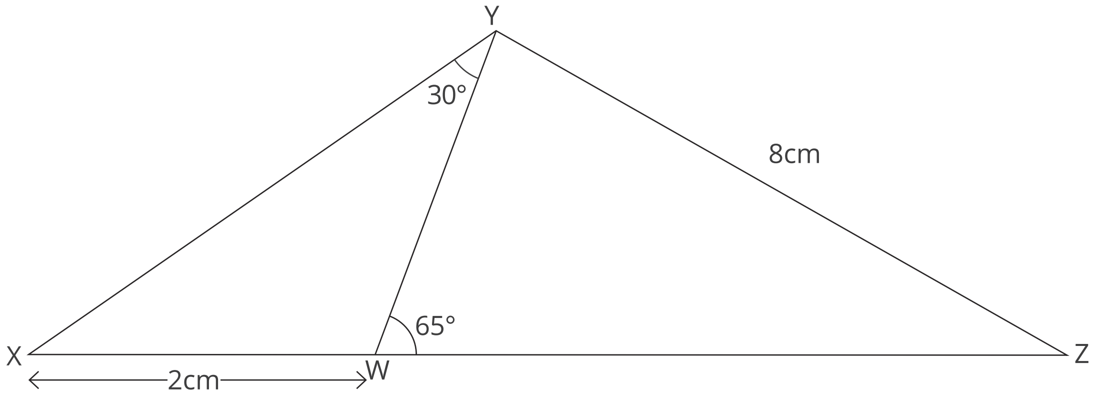


5. A plane takes off from Airport A, flies 200km due north, then turns on a bearing of 030° and flies a further 350km before landing at Airport B. How far is airport A from airport B in a straight line? Give your answer correct to the nearest kilometre.

6. Given that angle t is obtuse, find the size of angle x, giving your answer correct to 1d.p.



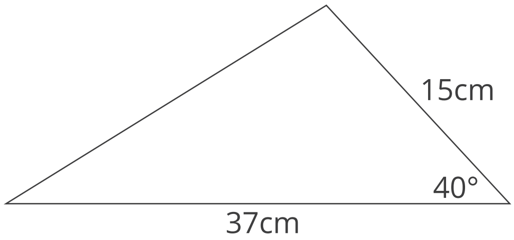
7. A triangle XYZ is divided by a line YW. Side YZ is 8cm, angle XYW is 30° and angle YWZ is 65°. Find the size of angle WZY, giving your answer correct to 3s.f.



For answers, go to page 124.

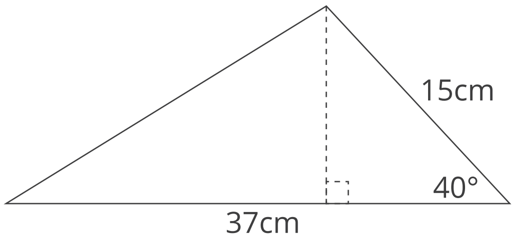
Area of Any Triangle

You can use the formula for the area of a triangle, Area = $\frac{1}{2}$ × base × height, to find the area of any triangle, but what if you don't know its height? Consider the triangle below:

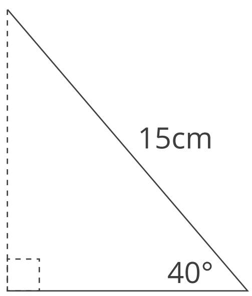


We know the length of two sides and the size of the angle between them. We don't know the height, but we have enough information to find it.

First, draw a perpendicular line from the base to the top of the triangle:



The length of this line is the height of the triangle. It is also one side of a right-angled triangle. We can use trigonometry ratios to find its length. This is our right-angled triangle:



We are given the length of the hypotenuse, 15cm, and an angle, 40°. We need to find the length of the opposite, so we'll use $\sin\theta = \frac{O}{H}$.

$\sin\theta = \frac{O}{H}$

$\sin(40°) = \frac{\text{height}}{15}$

height = 15 × sin(40°)

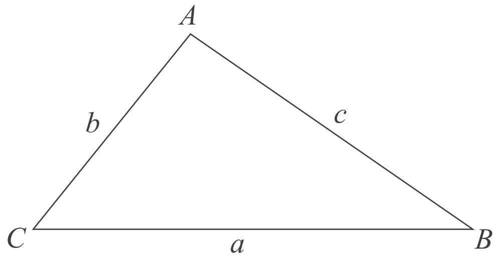
height = 9.64...cm

The height of our triangle is 9.64cm, so we can now find the area:

Area = $\frac{1}{2}$ × base × height:

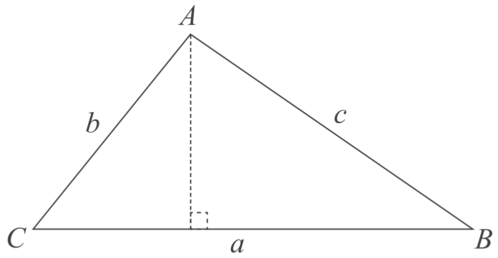
Area = $\frac{1}{2}$ × 37 × 9.64... = **178cm^2 (3s.f.)**

We can simplify the process above by finding a general rule. Consider the triangle ABC:



As with any non-right-angled triangle in trigonometry, we label the side opposite A as a, the side opposite B as b and the side opposite C as c.

For now, we will assume we are given angle C and sides a and b. Just as before, we will draw a perpendicular line from A:



This gives us a right-angled triangle. Again, we are given the hypotenuse (b) and an angle (C) and need to find the opposite, which is the height of triangle ABC.

Height = $b \times \sin C$

We now have a formula for the height of the triangle, and we know the length of the base of the triangle is a. Substituting both of these into the formula for the area of a triangle gives us:

Area = $\frac{1}{2} \times$ base $\times$ height

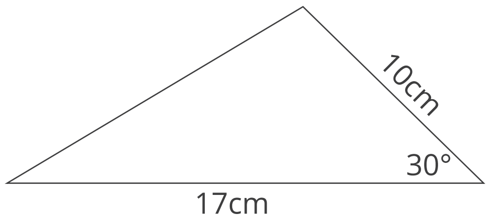
Area = $\frac{1}{2} \times a \times b \times \sin C$

Area = $\frac{1}{2} ab\sin C$

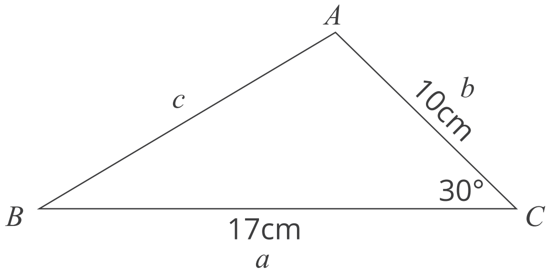
It's worth noting that the angle, C, must be enclosed by the two known sides. If it isn't, we need to use other methods to get the 3 measurements we need.

Example 1

Find the exact area of the triangle below.



Start by labelling the triangle. The angle we're given should be labelled C, but it doesn't matter which way we label a and b:



Now, we can find our area.

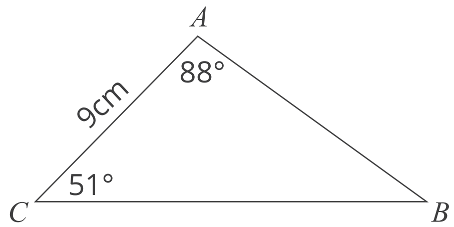
$$\text{Area} = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2} \times 17 \times 10 \times \sin(30°)$$

$$= \mathbf{42.5cm^2}$$

Example 2

Find the area of the triangle below.



In this case, we don't have enough information to use the formula straight away. When this is the case, we can usually use the sine rule to find another side (given two angles and a side) or the cosine rule to find an angle (given three sides, or two sides and a non-enclosed angle).

Cosine rule	**Sine rule**
$a^2 = b^2 + c^2 - 2bc\cos A$	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

We can't use the cosine rule as we would need to know 2 sides, but we can use the sine rule if we find angle B.

$B = 180 - (51 + 88)$

$= 41°$

Now, we can use the sine rule to find length a:

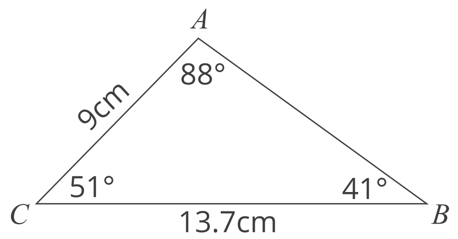
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin(88°)} = \frac{b}{\sin(41°)}$$

$$a = \frac{9}{\sin(41°)} \times \sin(88°)$$

$$= 13.7...\text{cm}$$

Let's add the measurements we've found to our diagram:



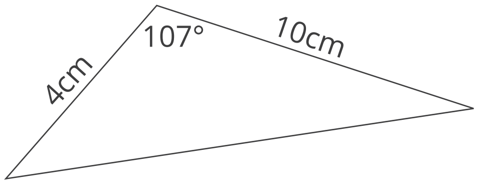
Now, we can use area = $\frac{1}{2}ab\sin C$

Area = $\frac{1}{2}$ × 13.7... × 9 × sin(51°) = **47.9cm^2 (3s.f.)**

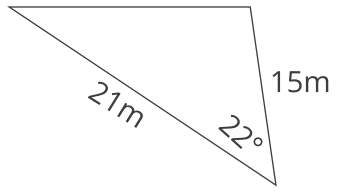
Your Turn:

1. Find the area of each of these triangles.

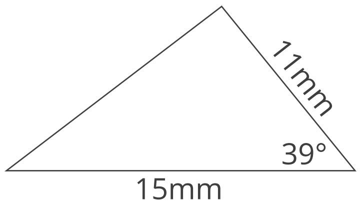
a.



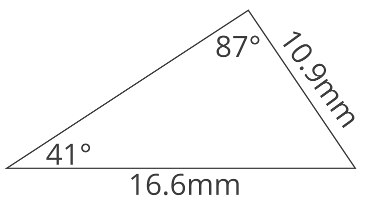
b.



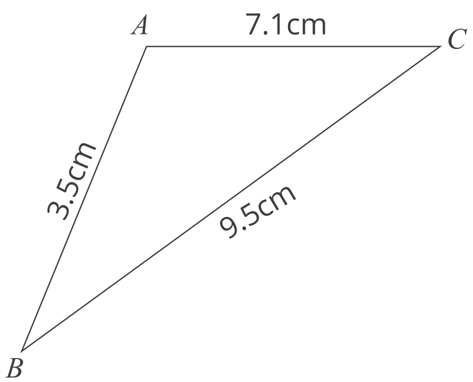
c.



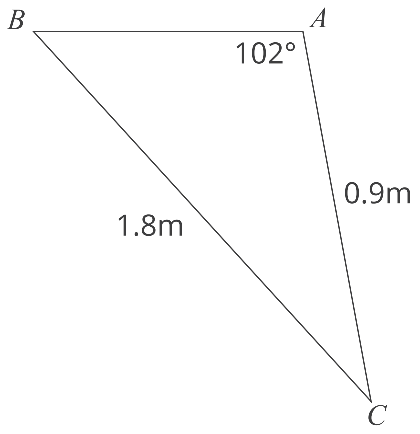
d.



2. By finding angle C, find the area of the triangle ABC.

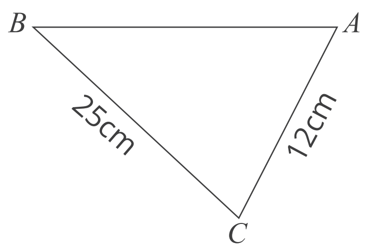


3. Find the area of the triangle below.

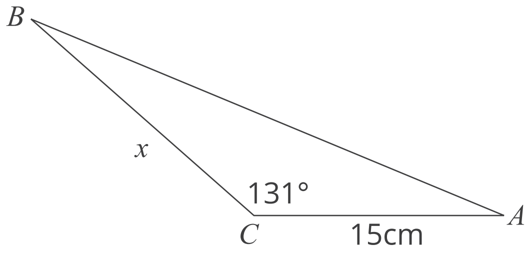


4. A ship patrols a dangerous area in the North Sea. The ship starts in port and travels 30km due North. It then travels 47km on a bearing on 135 before returning to port. Find the area enclosed by the ships journey.

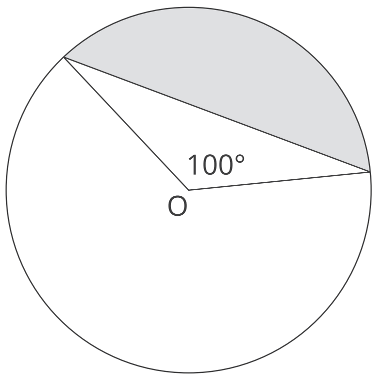
5. The area of the triangle below is 138cm^2. Find the size of angle C.



6. The area of the triangle below is 152cm^2. Find the length of the side x.



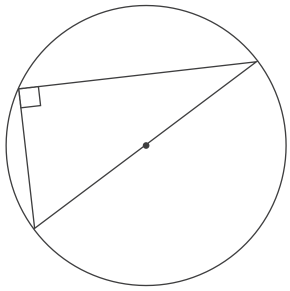
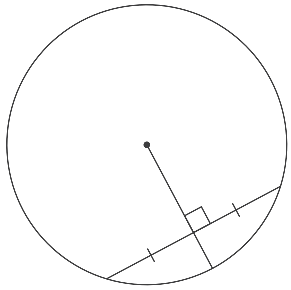
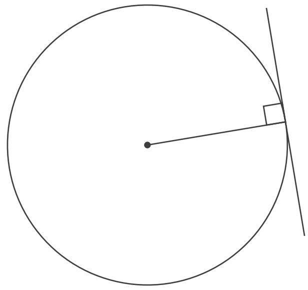
7. The shape below is a circle centre O and radius 4.5cm. Find the area of the shaded area below.



For answers, go to page 127.

Circle Theorems

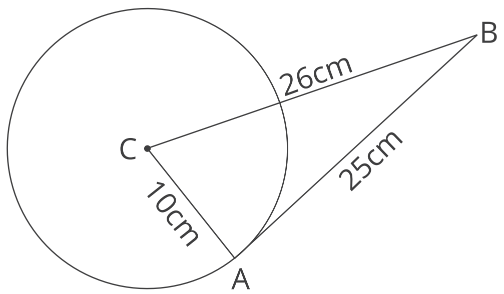
There are three circle theorems that are particularly helpful in A level maths.

		
The angle in a semicircle is a right angle.	The perpendicular from the centre to a chord bisects the chord.	The radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.

These will often be used as part of a larger, problem-solving question, so it is important that you are familiar with them. Unlike at GCSE, you don't usually have to give the circle theorem in the form of a reason. Instead, you are just expected to use the properties.

Example 1

The circle below has centre C and a point, A, lies on the circumference. Determine if the line AB is a tangent to the circle.



If AB is a tangent then the angle CAB will be a right angle. If this is the case, the triangle ABC is a right-angled triangle. We can check this by using Pythagoras' theorem - if the triangle is right-angled then $a^2 + b^2 = c^2$.

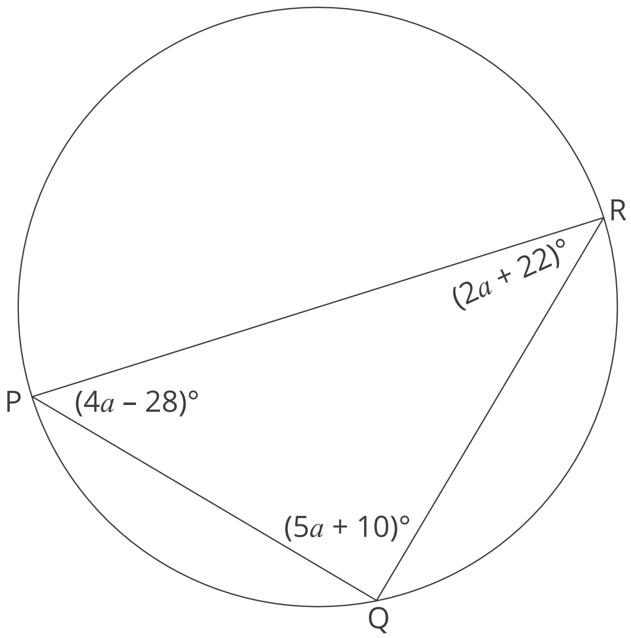
$$10^2 + 25^2 = 100 + 625$$
$$= 725$$

$$c = \sqrt{725} = 26.9\text{cm}$$

CAB is not a right angle, therefore AB is not a tangent.

Example 2

P, Q and R lie on the circumference of a circle. Show that PR is a diameter of the circle below.



If PR is a diameter, then the triangle PQR must be right-angled, with the right angle at Q. Let's find the value of a:

$(4a – 28) + (2a + 22) + (5a + 10) = 180$

$11a + 4 = 180$

$11a = 176$

$a = 16°$

Now we have a, let's substitute to find angle PQR.

PQR $= 5 \times 16 + 10$

$= 80 + 10$

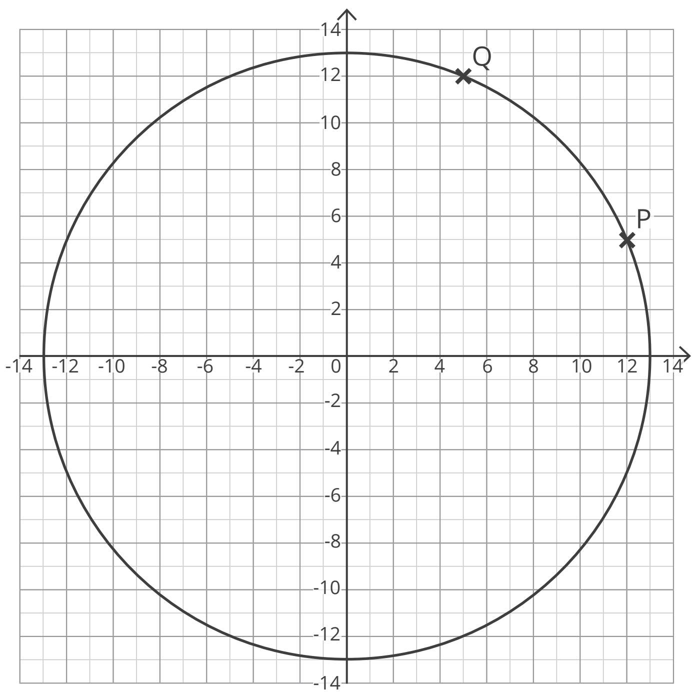
$= 90°$

Angle Q is a right angle, therefore PR is a diameter.

Example 3

The points P and Q have coordinates (12, 5) and (5, 12) respectively. P and Q sit on the circumference of a circle, C, centre (0, 0) and radius 13 units. Find the equation of the perpendicular bisector of PQ, then show that the perpendicular bisector is also a radius of the circle C. You cannot use any circle theorems.

As we are not given a diagram, we will start by sketching one.



To find the equation of the perpendicular bisector of PQ we need to know two things: the midpoint of PQ and the gradient of PQ.

Let's start by finding the midpoint. This is halfway between P and Q:

Midpoint = $\left(\frac{12 + 5}{2}, \frac{5 + 12}{2}\right)$

= (8.5, 8.5)

Now, let's find the gradient of PQ:

Gradient = $\frac{\text{change in } y}{\text{change in } x} = \frac{12 - 5}{5 - 12} = \frac{7}{-7} = -1$

The perpendicular line will therefore have a gradient of 1, as 1 is the negative reciprocal of -1.

We now have all the information to find the equation of the perpendicular bisector. The gradient is 1 and it passes through the point (8.5, 8.5). We can substitute this into $y = mx + c$.

$y = x + c$

$8.5 = 8.5 + c$

$c = 0$

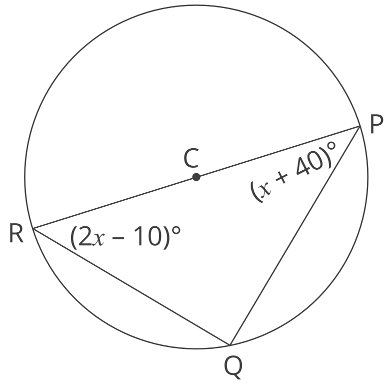
So, our equation is simply $\boldsymbol{y = x}$.

The last step is to show this is also the equation of a radius. **Any line that passes through the centre will be the equation of a radius, and the line $y = x$ clearly passes through the point (0, 0)**.

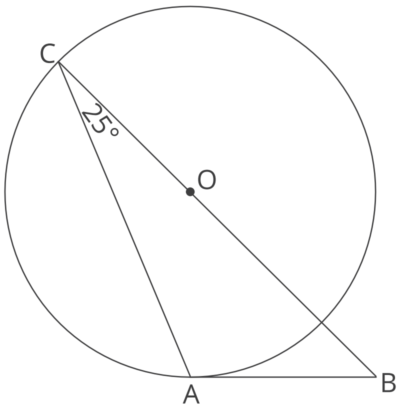
It's worth noting that this is not the equation of any radius; rather the equation of that specific radius. There are an infinite number of radiuses in the circle which will all have different equations, the only common factor is that they will all pass through the centre.

Your Turn:

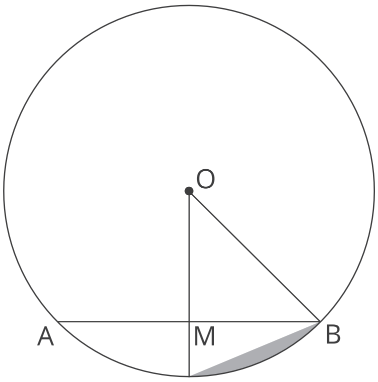
1. The points P, Q and R lie on the circumference of a circle, centre C. Find the value of x.



2. Below is a circle, centre O and radius 4cm. Given that AB is a tangent to the circle at point A, find the angle ABC.



3. The circle below has centre O and radius 10cm. Points A and B lie on the circumference, such that AB is 13.5cm long and OMB is a right angle.

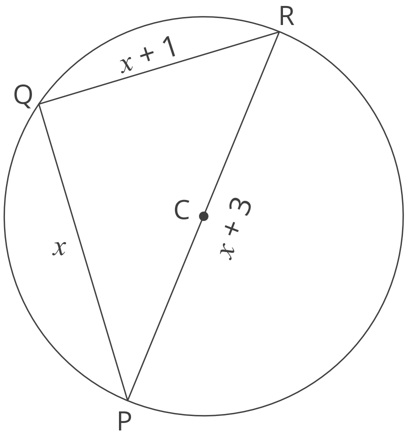


a. Find the angle OBM.

b. Find the area of the shaded region.

4. A circle, C, has centre (0, 0) and passes through the point (4, 1). Find the gradient of the tangent to C at the point (4, 1).

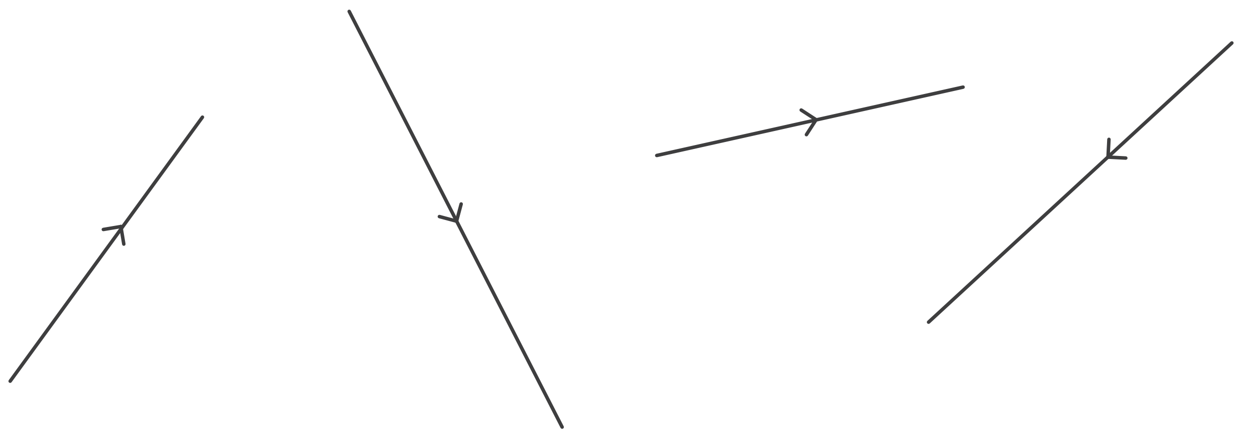
5. The points P, Q and R lie on the circumference of a circle, centre C. Find the exact area of the circle below. All measurements are given in centimetres.



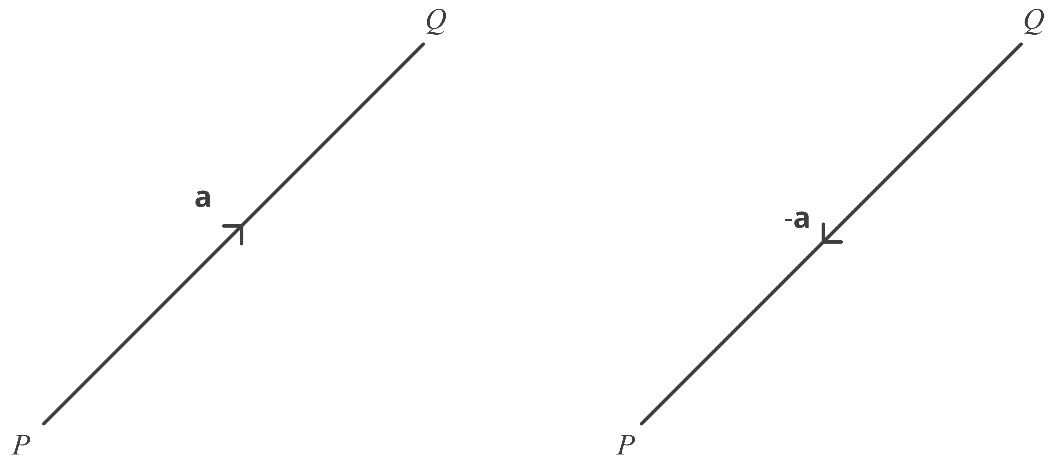
For answers, go to page 129.

Vectors

Vectors are useful in a number of areas of maths and physics. They describe a movement in a certain direction and of a certain magnitude. They are normally drawn as a directed line segment, with the arrow showing the direction.



We represent vectors in a few different ways. They can be written as a lower-case letter, (**a** or $\underline{a}$) or using the two defined end-points, with an arrow above to show direction. For example, the vector $\overrightarrow{PQ}$ is the vector from P to Q.



In the vectors above, $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are of equal magnitude but have opposite directions. This can be written as $\overrightarrow{PQ}$ = -$\overrightarrow{QP}$. We can also see that $\overrightarrow{PQ}$ = **a** and $\overrightarrow{QP}$ = -**a**.

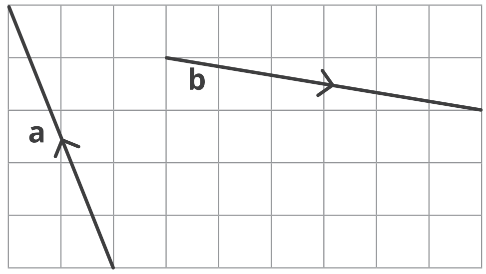
As well as labelling vectors, we also need a way to describe their size. We can do this using column vectors:

$$\binom{a}{b}$$

The top number, a, describes the movement in the x-direction – a movement to the right is positive while a movement to the left is negative. The bottom number, b, describes the movement in the y-direction – upwards is positive and downwards is negative.

Example 1

For the vectors **a** and **b** below, write **a** and **b** as column vectors. Each square in the diagram represents a single unit.

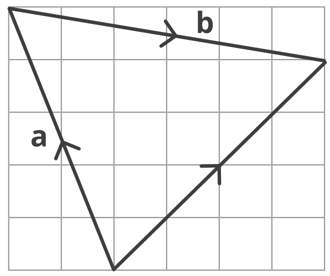


We can see that **a** is 2 units to the left and 5 units up: this is the column vector $\binom{-2}{5}$.

b is 6 units right and 1 unit down: this is the column vector $\binom{6}{-1}$.

We can also add and subtract vectors. You need to be able to do this diagrammatically (using diagrams) and algebraically. The outcome when adding two vectors is called the resultant.

Let's consider the vectors **a** and **b** from Example 1. To find **a** + **b** diagrammatically, we simply draw the two vectors connected. Adding **a** and **b** is essentially travelling along **a** then along **b**. As such, when drawing a diagram to represent an addition, it is important to make sure you can travel forwards along one vector and then forwards along the next. The resultant is the single vector from the start of the first vector to the end of the final vector.



We can see that the resultant vector here is $\binom{4}{4}$.

We could have found this vector without drawing a diagram, by adding the two column vectors. We do this by adding the top numbers of **a** and **b** to find the top number of the resultant, then doing the same for the bottom numbers:

$$\binom{-2}{5} + \binom{6}{-1} = \binom{-2+6}{5+-1} = \boldsymbol{\binom{4}{4}}$$

Example 2

Given that $\mathbf{p} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, find the vector $\mathbf{p} + \mathbf{q}$.

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= \begin{pmatrix} 6 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 + -2 \\ -1 + 3 \end{pmatrix} \\ &= \boldsymbol{\begin{pmatrix} 4 \\ 2 \end{pmatrix}} \end{aligned}$$

Given that we can add and subtract vectors, it is unsurprising that we can also multiply them by a scalar (a normal number). For example:

$2 \times \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

The vector $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ will be in the same direction as $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, but will have double the magnitude (it is twice as long).

Example 3

Given that $\mathbf{p} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, find the vectors below.

a. $3\mathbf{q}$	b. $\frac{1}{2}\mathbf{p}$	c. $2\mathbf{p} - 4\mathbf{q}$

a. $$\begin{aligned} 3\mathbf{q} &= 3 \times \begin{pmatrix} -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times -1 \\ 3 \times 5 \end{pmatrix} \\ &= \boldsymbol{\begin{pmatrix} -3 \\ 15 \end{pmatrix}} \end{aligned}$$

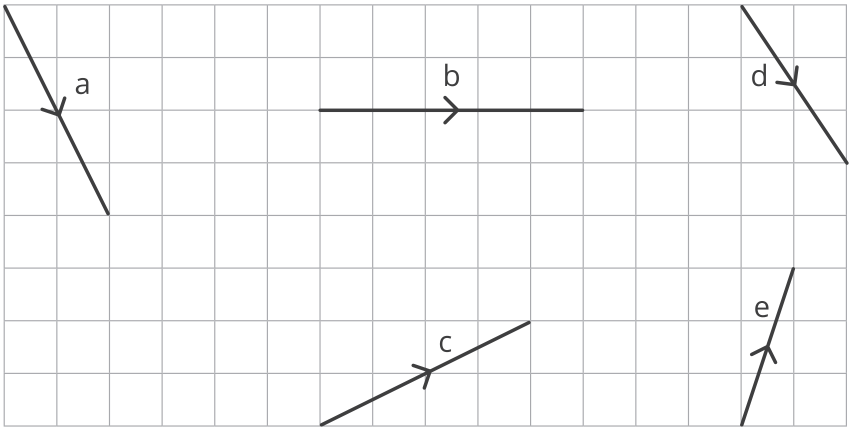
b. $$\begin{aligned} \tfrac{1}{2}\mathbf{p} &= \tfrac{1}{2} \times \begin{pmatrix} 2 \\ 10 \end{pmatrix} \\ &= \boldsymbol{\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \end{aligned}$$

c. $$\begin{aligned} 2\mathbf{p} - 4\mathbf{q} &= 2 \times \begin{pmatrix} 2 \\ 10 \end{pmatrix} - 4 \times \begin{pmatrix} -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 20 \end{pmatrix} - \begin{pmatrix} -4 \\ 20 \end{pmatrix} \\ &= \begin{pmatrix} 4 - -4 \\ 20 - 20 \end{pmatrix} \\ &= \boldsymbol{\begin{pmatrix} 8 \\ 0 \end{pmatrix}} \end{aligned}$$

It's really important you are careful with your negatives on questions like this – don't be afraid to double check a calculation on your calculator.

Your Turn:

1. Write each of the vectors below as a column vector.



a. ______________

b. ______________

c. ______________

d. ______________

e. ______________

2. Using the vectors in question 1, use diagrams to show each of these vector additions. You do not need to give the resultant vector.

a. **a** + **b**

c. **a** + **d** – **e**

b. **d** – **e**

d. **c** + 2**d**

3. For the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, write each of the following as column vectors.

a. $\mathbf{a} + \mathbf{b}$

e. $3\mathbf{a} - \mathbf{c}$

b. $\mathbf{a} - \mathbf{c}$

f. $\mathbf{c} - \frac{1}{2}\mathbf{b}$

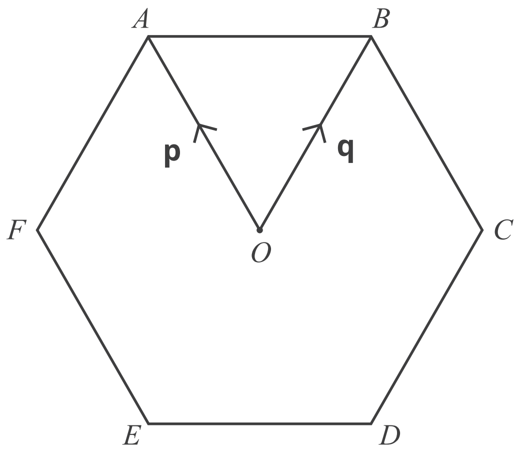
c. $\mathbf{a} - \mathbf{b} + \mathbf{c}$

g. $2\mathbf{a} - 3\mathbf{c}$

d. $2\mathbf{a}$

h. $\frac{1}{2}\mathbf{a} - 2\mathbf{b} + \frac{1}{4}\mathbf{c}$

This diagram is used for questions 4 and 5.



4. The diagram above shows a regular hexagon, centre O. Find the vectors below in terms of **p** and **q**.

   a. $\overrightarrow{AB}$

   b. $\overrightarrow{ED}$

   c. $\overrightarrow{OE}$

   d. $\overrightarrow{DA}$

   e. $\overrightarrow{AF}$

   f. $\overrightarrow{BF}$

5. Given that M is the midpoint of BC, find the vector $\overrightarrow{EM}$ in terms of **p** and **q**.

For answers, go to page 131.

Extra Space