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A Level Mathematics

Induction Pack

ANSWERS

Negative and Zero Powers (Answers)

1. a. What is $3^5 \div 3^2$ in index form?

   $\mathbf{3^3}$

   b. What is $3^2 \div 3^2$ in index form?

   $\mathbf{3^0}$

   c. Evaluate $3^2 \div 3^2$

   1

   d. Evaluate 3^0

   1

   e. Evaluate 27.54^0

   1

   f. Evaluate $2.7523^0 \times 268^1 \times 892^0$

   268

2. Evaluate the following:

   a. 5^{-2} — $\mathbf{\frac{1}{25}}$

   b. 8^{-2} — $\mathbf{\frac{1}{64}}$

   c. 3^{-3} — $\mathbf{\frac{1}{27}}$

   d. 2^{-5} — $\mathbf{\frac{1}{32}}$

3. Write each in index form:

   a. $\frac{1}{16}$ — $\mathbf{4^{-2}}$

   b. $\frac{1}{49}$ — $\mathbf{7^{-2}}$

   c. $\frac{1}{125}$ — $\mathbf{5^{-3}}$

   d. $\frac{1}{1000}$ — $\mathbf{10^{-3}}$

4. Evaluate, giving your answers as fractions in their simplest form:

   a. $(\frac{3}{5})^{-1}$ — $\mathbf{\frac{5}{3}}$

   b. $(\frac{7}{8})^{-2}$ — $\mathbf{\frac{64}{49}}$

   c. $(\frac{1}{4})^{-3}$ — $\mathbf{\frac{64}{1} = 64}$

   d. $(\frac{2}{3})^{-3}$ — $\mathbf{\frac{27}{8}}$

5. Evaluate, giving your answers as fractions in their simplest form:

   a. $(3x)^{-2}$ — $\mathbf{\frac{1}{9x^2}}$

   b. $(2x^3)^{-2}$ — $\mathbf{\frac{1}{4x^6}}$

   c. $(5x^4)^{-3}$ — $\mathbf{\frac{1}{125x^{12}}}$

   d. $(2x^2y^3)^{-4}$ — $\mathbf{\frac{1}{16x^8y^{12}}}$

Fractional Powers (Answers)

1. Evaluate the following:

   a. $36^{\frac{1}{2}}$ b. $1000^{\frac{1}{3}}$ c. $64^{\frac{1}{3}}$ d. $81^{-\frac{1}{2}}$

   a. $\mathbf{\sqrt{36} = 6}$ b. $\mathbf{\sqrt[3]{1000} = 10}$ c. $\mathbf{\sqrt[3]{64} = 4}$ d. $\mathbf{\frac{1}{\sqrt{81}} = \frac{1}{9}}$

2. Evaluate the following:

   a. $27^{\frac{2}{3}}$ b. $8^{\frac{4}{3}}$ c. $49^{\frac{3}{2}}$ d. $64^{\frac{2}{3}}$

   a. $\mathbf{= \sqrt[3]{27^2}}$ b. $\mathbf{= \sqrt[3]{8^4}}$ c. $\mathbf{= \sqrt{49^3}}$ d. $\mathbf{= \sqrt[3]{64^2}}$

   a. $\mathbf{= 3^2}$ b. $\mathbf{= 2^4}$ c. $\mathbf{= 7^3}$ d. $\mathbf{= 4^2}$

   a. $\mathbf{= 9}$ b. $\mathbf{= 16}$ c. $\mathbf{= 343}$ d. $\mathbf{= 16}$

3. Express in the form $a^{\frac{m}{n}}$, where m and n are integers.

   a. $\sqrt{a^3}$

   $\boldsymbol{a^{\frac{3}{2}}}$

   b. $\sqrt[3]{a^5}$

   $\boldsymbol{a^{\frac{5}{3}}}$

   c. $\frac{1}{\sqrt{a^7}}$

   $\boldsymbol{a^{-\frac{7}{2}}}$

   d. $\sqrt{a} \times \frac{1}{\sqrt{a^5}}$

   $\boldsymbol{a^{\frac{1}{2}} \times a^{-\frac{5}{2}}}$

   $\boldsymbol{= a^{\frac{1}{2} + -\frac{5}{2}}}$

   $\boldsymbol{= a^{-\frac{4}{2}}}$

   or simplify to $\boldsymbol{a^{-2}}$

4. Write the following expressions in order, from smallest to largest:

   $25^{\frac{1}{2}}$, $8^{\frac{2}{3}}$, $27^{\frac{1}{3}}$, $(\frac{1}{9})^{-\frac{3}{2}}$, $(\frac{1}{12})^{-1}$, $(27^{\frac{5}{3}})^0$,

   $\mathbf{25^{\frac{1}{2}} = 5}$, $\mathbf{8^{\frac{2}{3}} = 4}$, $\mathbf{27^{\frac{1}{3}} = 3}$, $\mathbf{(\frac{1}{9})^{-\frac{3}{2}} = 27}$, $\mathbf{(\frac{1}{12})^{-1} = 12}$, $\mathbf{(27^{\frac{5}{3}})^0 = 1}$

   $\mathbf{(27^{\frac{5}{3}})^0}$, $\mathbf{27^{\frac{1}{3}}}$, $\mathbf{8^{\frac{2}{3}}}$, $\mathbf{25^{\frac{1}{2}}}$, $\mathbf{(\frac{1}{12})^{-1}}$, $\mathbf{(\frac{1}{9})^{-\frac{3}{2}}}$

5. Write $64^{\frac{2}{3}} \times 2^3$ in the form 2^a, where a is a positive integer.

$64 = 2^6$

$64^{\frac{2}{3}} = (2^6)^{\frac{2}{3}}$

$= 2^{\frac{12}{3}}$ **(use the index law for brackets)**

$= 2^4$

$64^{\frac{2}{3}} \times 2^3 = 2^4 \times 2^3$

$= 2^7$ **(use the index law for multiplication)**

Index Laws (Answers)

1. Simplify each expression. Give your answers in index form.

a. $5^4 \times 5^8$

$\mathbf{5^{4+8} = 5^{12}}$

b. $m^4 \div m^2$

$\boldsymbol{m^{4-2} = m^2}$

c. $(a^3)^2$

$\boldsymbol{a^{3\times 2} = a^6}$

d. $3^5 \times 3$

$\mathbf{3^{5+1} = 3^6}$

2. Simplify each expression. Give your answers in index form.

a. $3^8 \times 3^{-2}$

$\mathbf{3^{8+-2} = 3^6}$

b. $\dfrac{h^{-3}}{h^5}$

$\boldsymbol{h^{-3-5} = h^{-8}}$

c. $p^{-2} \div p^{-9}$

$\boldsymbol{p^{-2--9} = p^7}$

d. $(5^{-3})^{-2}$

$\mathbf{5^{-3\times -2} = 5^6}$

3. Simplify each expression. Give your answers in index form.

a. $3a^2 \times 3a^5$

$\boldsymbol{9a^7}$

b. $(3x^4)^3$

$\boldsymbol{27x^{12}}$

c. $\dfrac{12x^3}{4x^5}$

$\boldsymbol{3x^{-2}}$

d. $a^2b^5 \times a^4b^{-8}$

$\boldsymbol{a^6b^{-3}}$

4. Simplify the expression. Give your answer in index form.

$$\left(\frac{3a^5 \times 6a^{-7}}{2a^5}\right)^2 = \left(\frac{18a^{-2}}{2a^5}\right)^2$$

$$= (9a^{-7})^2$$

$$= 81a^{-14}$$

Simplifying Surds (Answers)

Multiplying and Dividing Surds

1. a. $\sqrt{5} \times \sqrt{7}$

$\sqrt{35}$

b. $3\sqrt{2} \times 4\sqrt{5}$

$12\sqrt{10}$

c. $\sqrt{15} \div \sqrt{3}$

$\sqrt{5}$

d. $18\sqrt{20} \div 6\sqrt{5}$

$3\sqrt{4} = 3 \times 2 = 6$

e. $5\sqrt{2} \times 3\sqrt{8}$

$15\sqrt{16}$

$= 15 \times 4 = 60$

f. $2\sqrt{3} \times 5$

$10\sqrt{3}$

g. $(5\sqrt{3})^2$

$25 \times 3 = 75$

h. $(2\sqrt{5})^3$

$2^3 \times (\sqrt{5})^3$

$= 8 \times 5\sqrt{5} = 40\sqrt{5}$

2. A right-angled triangle has a height of $6\sqrt{5}$cm and a base of $7\sqrt{3}$cm. Find its area.

$6\sqrt{5} \times 7\sqrt{3} \div 2$

$= 42\sqrt{15} \div 2$

$= 21\sqrt{15}\text{cm}^2$

Addition and Subtraction of Surds

1. Simplify these surds (remember: the key is to find a square factor).

a. $\sqrt{20}$

$\sqrt{4}\,\sqrt{5} = 2\sqrt{5}$

b. $\sqrt{48}$

$\sqrt{16}\,\sqrt{3} = 4\sqrt{3}$

c. $\sqrt{75}$

$\sqrt{25}\,\sqrt{3} = 5\sqrt{3}$

d. $5\sqrt{8}$

$5(2\sqrt{2}) = 10\sqrt{2}$

2. Give your answers in the form $a\sqrt{b}$

a. $\sqrt{2} + \sqrt{18}$

$\mathbf{\sqrt{2} + 3\sqrt{2} = 4\sqrt{2}}$

b. $\sqrt{50} - \sqrt{200}$

$\mathbf{\sqrt{25}\,\sqrt{2} - \sqrt{100}\,\sqrt{2}}$

$\mathbf{= 5\sqrt{2} - 10\sqrt{2} = -5\sqrt{2}}$

c. $4\sqrt{80} + 3\sqrt{45}$

$\mathbf{4\sqrt{16}\,\sqrt{5} + 3\sqrt{9}\,\sqrt{5}}$

$\mathbf{= 16\sqrt{5} + 9\sqrt{5} = 25\sqrt{5}}$

d. $2\sqrt{50} + 5\sqrt{32}$

$\mathbf{2\sqrt{25}\,\sqrt{2} + 5\sqrt{16}\,\sqrt{2}}$

$\mathbf{= 10\sqrt{2} + 20\sqrt{2} = 30\sqrt{2}}$

3. A rectangle has a width of $6\sqrt{75}$m and a height of $2\sqrt{12}$m. What is its perimeter?

$\mathbf{\sqrt{75} = 5\sqrt{3}}$

$\mathbf{\sqrt{12} = 2\sqrt{3}}$

$\mathbf{6\sqrt{75} + 6\sqrt{75} + 2\sqrt{12} + 2\sqrt{12}}$

$\mathbf{= 12\sqrt{75} + 4\sqrt{12}}$

$\mathbf{= 60\sqrt{3} + 8\sqrt{3} = 68\sqrt{3}m}$

Challenge

A right-angled triangle has a base of $2\sqrt{18}$cm and a height of $2\sqrt{32}$cm. Find the perimeter of the triangle.

Using Pythagoras to find the hypotenuse:

$\mathbf{\sqrt{(2\sqrt{18})^2 + (2\sqrt{32})^2}}$

$\mathbf{= \sqrt{72 + 128}}$

$\mathbf{= \sqrt{200}\ cm}$

Add the three sides:

$\mathbf{2\sqrt{18} + 2\sqrt{32} + \sqrt{200}}$

$\mathbf{= 6\sqrt{2} + 8\sqrt{2} + 10\sqrt{2} = 24\sqrt{2}cm}$

Rationalising the Denominator (Answers)

1. Rationalise the denominator of each fraction.

a. $\frac{5}{\sqrt{2}}$

$\mathbf{\frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}}$

b. $\frac{4}{\sqrt{3}}$

$\mathbf{\frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}}$

c. $\frac{\sqrt{2}}{\sqrt{3}}$

$\mathbf{\frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}}$

d. $\frac{3}{2\sqrt{5}}$

$\mathbf{\frac{3}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{10}}$

e. $\frac{2-\sqrt{3}}{\sqrt{3}}$

$\mathbf{\frac{2-\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}-3}{3}}$

2. Rationalise the denominator of $\frac{\sqrt{5}}{\sqrt{80}}$. Give your answer as a fraction in its simplest form.

$\mathbf{\frac{\sqrt{5}}{\sqrt{80}}}$

$\mathbf{= \frac{\sqrt{5}}{\sqrt{80}} \times \frac{\sqrt{80}}{\sqrt{80}}}$

$\mathbf{= \frac{\sqrt{400}}{80}}$

$\mathbf{= \frac{20}{80}}$

$\mathbf{= \frac{1}{4}}$

Or: $\mathbf{\frac{\sqrt{5}}{\sqrt{80}}}$

$\mathbf{= \frac{\sqrt{5}}{\sqrt{16}\sqrt{5}}}$

$\mathbf{= \frac{\sqrt{5}}{4\sqrt{5}}}$

$\mathbf{= \frac{1}{4}}$

3. What is $\frac{2\sqrt{2}}{\sqrt{6}} + \frac{1}{\sqrt{3}}$? Give your answer in its simplest terms.

$\mathbf{\frac{2\sqrt{2}}{\sqrt{6}}}$

$\mathbf{= \frac{2\sqrt{2}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}}$

$\mathbf{= \frac{2\sqrt{12}}{6}}$

$\mathbf{= \frac{\sqrt{12}}{3}}$

$\mathbf{= \frac{2\sqrt{3}}{3}}$

$\mathbf{\frac{1}{\sqrt{3}}}$

$\mathbf{= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}}$

$\mathbf{= \frac{\sqrt{3}}{3}}$

Therefore, $\mathbf{\frac{2\sqrt{2}}{\sqrt{6}} + \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}$

$\mathbf{= \frac{3\sqrt{3}}{3}}$

$\mathbf{= \sqrt{3}}$

4. a. $\frac{5}{\sqrt{2} + 7}$

$\frac{5}{\sqrt{2} + 7} \times \frac{\sqrt{2} - 7}{\sqrt{2} - 7}$

$= \frac{5(\sqrt{2} - 7)}{(\sqrt{2} + 7)(\sqrt{2} - 7)}$

$= \frac{35 - 5\sqrt{2}}{47}$ **or equivalent**

b. $\frac{1}{\sqrt{5} - 3}$

$\frac{1}{\sqrt{5} - 3} \times \frac{\sqrt{5} + 3}{\sqrt{5} + 3}$

$= \frac{-\sqrt{5} - 3}{4}$ **or equivalent**

c. $\frac{1 + \sqrt{2}}{\sqrt{3} + 2}$

$\frac{1 + \sqrt{2}}{\sqrt{3} + 2} \times \frac{\sqrt{3} - 2}{\sqrt{3} - 2}$

$= \frac{(1 + \sqrt{2})(\sqrt{3} - 2)}{(\sqrt{3} + 2)(\sqrt{3} - 2)}$

$= \frac{\sqrt{3} - 2 + \sqrt{6} - 2\sqrt{2}}{(\sqrt{3} + 2)(\sqrt{3} - 2)}$

$= \frac{\sqrt{3} - 2 + \sqrt{6} - 2\sqrt{2}}{3 - 4}$

$= 2 + 2\sqrt{2} - \sqrt{3} - \sqrt{6}$

or equivalent

Challenge

Amy is laying tiles in her rectangular bathroom. By the time she has finished, she has used 8m^2 worth of tiles. She knows the length of one side of the room is $(\sqrt{5} + 2)$m but, unfortunately, she has lost her tape measure. Amy still needs to work out the perimeter of the room. Calculate the perimeter of the room, giving your answer in its simplest form.

Missing length = $\frac{8}{\sqrt{5} + 2}$

$= \frac{8\sqrt{5} - 16}{1}$

$= (8\sqrt{5} - 16)\text{m}$

Perimeter $= 2(8\sqrt{5} - 16) + 2(\sqrt{5} + 2)$

$= 16\sqrt{5} - 32 + 2\sqrt{5} + 4$

$= (18\sqrt{5} - 28)\text{m}$

Expanding Polynomials (Answers)

Expanding (Multiplying) Brackets

Expand and simplify:

1. $5(2x - 7)$

   $10x - 35$

2. $8x(2x + 3)$

   $16x^2 + 24x$

3. $7a(3a + 2b - 4)$

   $21a^2 + 14ab - 28a$

4. $5(2x + 1) + 3(x + 4)$

   $10x + 5 + 3x + 12 = 13x + 17$

5. $8y(y - 4) - 2y(3 - y)$

   $8y^2 - 32y - 6y + 2y^2 = 10y^2 - 38y$

   Remember that a negative times a negative gives a positive!

6. $(3x + 2)(x + 5)$

   $3x^2 + 15x + 2x + 10 = 3x^2 + 17x + 10$

7. $(x - 4)(3x - 9)$

   $3x^2 - 9x - 12x + 36 = 3x^2 - 21x + 36$

8. $(a + b)(b - c)$

   $ab - ac + b^2 - bc$

   Note that it is convention to write expressions in decreasing powers, so this might be written instead as $b^2 + ab - ac - bc$

9. $(3x + 2)^2$

   $\mathbf{(3x + 2)(3x + 2) = 9x^2 + 6x + 6x + 4}$

   $\mathbf{= 9x^2 + 12x + 4}$

   A common mistake is to simply square both terms. To prevent this, you must write the whole expression out before expanding.

10. $(x + 8)(2x + y - 4)$

    $\mathbf{2x^2 + xy - 4x + 16x + 8y - 32 = 2x^2 + xy + 12x + 8y - 32}$

11. $(x + 3)(x + 4)(x + 1)$

    $\mathbf{(x^2 + 7x + 12)(x + 1) = x^3 + 8x^2 + 19x + 12}$

12. $(2x - 5)(x - 2)(x + 7)$

    $\mathbf{(2x^2 - 9x + 10)(x + 7) = 2x^3 + 5x^2 - 53x + 70}$

13. $(x + 1)^3$

    $\mathbf{(x + 1)(x + 1)(x + 1) = (x^2 + 2x + 1)(x + 1)}$

    $\mathbf{= x^3 + 3x^2 + 3x + 1}$

14. $(x + 2)^2(x + 5)$

    $\mathbf{(x + 2)(x + 2)(x + 5) = (x^2 + 4x + 4)(x + 5)}$

    $\mathbf{= x^3 + 9x^2 + 24x + 20}$

Factorising

Factorise fully:

1. $12x + 15$

   $\mathbf{3(4x + 5)}$

2. $27x - 18$

   $\mathbf{9(3x - 2)}$

3. $10y^2 + 28y$

   $2y(5y + 14)$

4. $14ab + 21a$

   $7a(2b + 3)$

5. $32x + 40y - 24$

   $8(4x + 5y - 3)$

6. $10x^2y - 15xy^2$

   $5xy(2x - 3y)$

7. $12a^3b^2 + 18a^2b^3 - 27ab^4$

   $3ab^2(4a^2 + 6ab - 9b^2)$

8. $a(b + c) + 5(b + c)$

   $(a + 5)(b + c)$

9. $x(y + 3) + 2(y + 3)$

   $(x + 2)(y + 3)$

10. $2r(a - 4) - p(a - 4)$

    $(2r - p)(a - 4)$

Factorising Quadratic Expressions (Answers)

Factorising: When a = 1

Factorise fully:

1. $x^2 + 7x + 10$

   $(x + 2)(x + 5)$

2. $x^2 + 12x + 20$

   $(x + 10)(x + 2)$

3. $x^2 + 4x - 21$

   $(x + 7)(x - 3)$

4. $x^2 - x - 6$

   $(x + 2)(x - 3)$

   The coefficient of x in this question is -1.

5. $x^2 - 13x + 30$

   $(x - 10)(x - 3)$

6. $x^2 - 10x + 25$

   $(x - 5)(x - 5)$

   This could also be written as $(x - 5)^2$.

Factorising: The Difference of Two Squares

Factorise fully:

1. $x^2 - 36$

   $(x + 6)(x - 6)$

2. $a^2 - 81$

   $(a + 9)(a - 9)$

3. $4x^2 - 9$

   $(2x + 3)(2x - 3)$

4. $25a^2 - b^2$

   $(5a + b)(5a - b)$

5. $9x^2 - 100y^2$

   $(3x + 10y)(3x - 10y)$

6. $x^4 - y^2$

   $(x^2 + y)(x^2 - y)$

Factorising – When $a \neq 1$

Factorise fully:

1. $2x^2 + 11x + 12$

$2 \times 12 = 24$

$8 \times 3 = 24$ and $8 + 3 = 11$

$$2x^2 + 11x + 12 = 2x^2 + 8x + 3x + 12$$
$$= 2x(x + 4) + 3(x + 4)$$
$$= (x + 4)(2x + 3)$$

2. $3x^2 + 26x + 35$

$3 \times 35 = 105$

$5 \times 21 = 105$ and $5 + 21 = 26$

$$3x^2 + 26x + 35 = 3x^2 + 5x + 21x + 35$$
$$= x(3x + 5) + 7(3x + 5)$$
$$= (x + 7)(3x + 5)$$

3. $4x^2 + 8x - 21$

$4 \times -21 = -84$

$14 \times -6 = -84$ and $14 + -6 = 8$

$$4x^2 + 8x - 21 = 4x^2 + 14x - 6x - 21$$
$$= 2x(2x + 7) - 3(2x + 7)$$
$$= (2x + 7)(2x - 3)$$

4. $3x^2 - 19x + 20$

$3 \times 20 = 60$

$-4 \times -15 = 60$ and $-4 + -15 = -19$

$$3x^2 - 19x + 20 = 3x^2 - 4x - 15x + 20$$
$$= x(3x - 4) - 5(3x - 4)$$
$$= (3x - 4)(x - 5)$$

Notice that the common factor for the second pair of expressions needed to be -5 so that the expressions inside the brackets matched.

Completing the Square

Write each equation in completed square form, and then find the coordinates of the turning point.

1. $y = x^2 + 8x + 23$

   $\mathbf{y = (x + 4)^2 + 7}$

   (-4, 7)

2. $y = x^2 - 6x + 1$

   $\mathbf{y = (x - 3)^2 - 8}$

   (3, -8)

3. $y = x^2 + 4x - 6$

   $\mathbf{y = (x + 2)^2 - 10}$

   (-2, -10)

4. $y = x^2 + 3x + 9$

   $\mathbf{y = (x + 1.5)^2 + 6.75}$ **(or fractional equivalent)**

   (-1.5, 6.75)

5. $y = x^2 - 5x - 8$

   $\mathbf{y = (x - 2.5)^2 - 14.25}$ **(or fractional equivalent)**

   (2.5, -14.25)

6. $y = 2x^2 + 12x + 7$

   $\mathbf{y = 2(x^2 + 6x) + 7}$

   $\mathbf{y = 2((x + 3)^2 - 9) + 7}$

   $\mathbf{y = 2(x + 3)^2 - 11}$

   (-3, -11)

7. $y = 3x^2 + 12x + 2$

   $\mathbf{y = 3(x^2 + 4x) + 2}$

   $\mathbf{y = 3((x + 2)^2 - 4) + 2}$

   $\mathbf{y = 3(x + 2)^2 -10}$

   (-2, -10)

8. $y = 2x^2 + 6x + 23$

   $\mathbf{y = 2(x^2 + 3x) + 23}$

   $\mathbf{y = 2((x + 1.5)^2 - 2.25) + 23}$

   $\mathbf{y = 2(x + 1.5)^2 + 18.5}$

   (-1.5, 18.5)

Linear Equations and Inequalities (Answers)

1. Solve the following equations:

   a. $8(2x + 3) = 24$

   $\mathbf{2x + 3 = 3}$

   $\mathbf{2x = 0}$

   $\mathbf{x = 0}$

   d. $4(2x - 5) = 3(x + 2)$

   $\mathbf{8x - 20 = 3x + 6}$

   $\mathbf{5x = 26}$

   $\mathbf{x = \frac{26}{5}}$

   b. $\frac{3x - 4}{2} = 5$

   $\mathbf{3x - 4 = 10}$

   $\mathbf{3x = 14}$

   $\mathbf{x = \frac{14}{3}}$

   e. $\frac{5x - 7}{x} = 9$

   $\mathbf{5x - 7 = 9x}$

   $\mathbf{-7 = 4x}$

   $\mathbf{x = -\frac{7}{4}}$

   c. $2(\frac{3(x + 1)}{5}) = 6$

   $\mathbf{\frac{3(x + 1)}{5} = 3}$

   $\mathbf{3x + 3 = 15}$

   $\mathbf{3x = 12}$

   $\mathbf{x = 4}$

   f. $8 - \frac{3x}{2 + x} = 10$

   $\mathbf{-\frac{3x}{2 + x} = 2}$

   $\mathbf{-3x = 4 + 2x}$ **(or equivalent)**

   $\mathbf{-5x = 4}$

   $\mathbf{x = -\frac{4}{5}}$

2. Solve the following inequalities:

   a. $8x + 3 > 2(x + 5)$

   $\mathbf{8x + 3 > 2x + 10}$

   $\mathbf{6x > 7}$

   $\mathbf{x > \frac{7}{6}}$

   b. $\frac{2x - 1}{7} \leq 3$

   $\mathbf{2x - 1 \leq 21}$

   $\mathbf{2x \leq 22}$

   $\mathbf{x \leq 11}$

c. $7 \le 4x + 5 < 19$

$2 \le 4x < 14$

$\frac{1}{2} \le x < \frac{7}{2}$

d. $5(3 - 2x) \ge 1$

$15 - 10x \ge 1$

$-10x \ge -14$

$x \le \frac{7}{5}$

3. Find the set of solutions which satisfies the following inequalities:
   $8x \ge 5 - x$ and $-4 < 3x + 1 \le 10$

   Solving the first:

   $9x \ge 5$

   $x \ge \frac{5}{9}$

   Solving the second:

   $-5 < 3x \le 9$

   $-\frac{5}{3} < x \le 3$

   The set of solutions which satisfies both is $\frac{5}{9} \le x \le 3$

Graphs (Answers)

Equations of Straight-Line Graphs / Parallel and Perpendicular Lines

1. Complete the table:

Equation	Gradient	y-intercept
$y = 3x + 7$	**3**	**7**
$y = 2 - x$	**-1**	**2**
$2y = x + 5$	$\mathbf{\frac{1}{2}}$ **or 0.5**	$\mathbf{\frac{5}{2}}$ **or 2.5**
$3y + 2x = -1$	$\mathbf{-\frac{2}{3}}$	$\mathbf{-\frac{1}{3}}$
$\mathbf{y = -2x}$	-2	0
$\mathbf{y = \frac{3}{4}x - 1}$	$\frac{3}{4}$	-1

2. Find the equation of the line passing through the points with coordinates (2, 3) and (4, 1).

$m = \frac{1 - 3}{4 - 2} = -1$

$y = -x + c$

$3 = -2 + c$

$c = 5$

$y = -x + 5$

3. A line whose gradient is $\frac{1}{3}$ passes through the point (-6, 9). Work out the equation of this line, giving your answer in the form $ay + bx = c$, where a, b and c are integers.

$y = \frac{1}{3}x + c$

$9 = \frac{1}{3} \times -6 + c$

$9 = -2 + c$

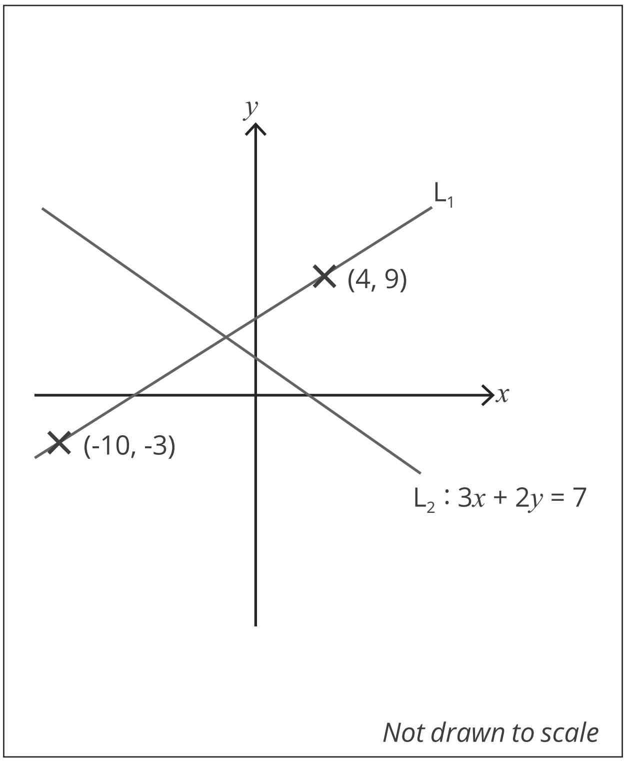
$c = 11$

$y = \frac{1}{3}x + 11$

$3y = x + 33$

$3y - x = 33$

4. The diagram shows two straight lines. Are the lines perpendicular? Justify your answer.



Let m_1 be the gradient of L_1 and m_2 be the gradient of L_2.

$$m_1 = \frac{9 - -3}{4 - -10}$$

$$= \frac{12}{14}$$

$$= \frac{6}{7}$$

Rearrange the equation for L_2 to make y the subject.

$$y = -\frac{3}{2}x + \frac{7}{2}$$

$$m_2 = -\frac{3}{2}$$

$\frac{6}{7} \times -\frac{3}{2} \neq -1$ so the lines are not perpendicular.

5. Does the line with equation $2x + 5y = -1$ pass through the point with coordinates (2, -1)?

Substitute $x = 2$ and $y = -1$ into the expression $2x + 5y$.

$2 \times 2 + 5 \times (-1) = -1$

Since this is equal to -1, as in the original equation, the line must pass through this point.

Quadratic Graphs

1. Consider the curve with equation $y = x^2 + 4x - 5$.

   a. Find the coordinates of the point where this curve intersects the y-axis.

   $y = 0^2 + 4 \times 0 - 5 = -5$

   (0, -5)

   b. Find the coordinates of the points where this curve intersects the x-axis.

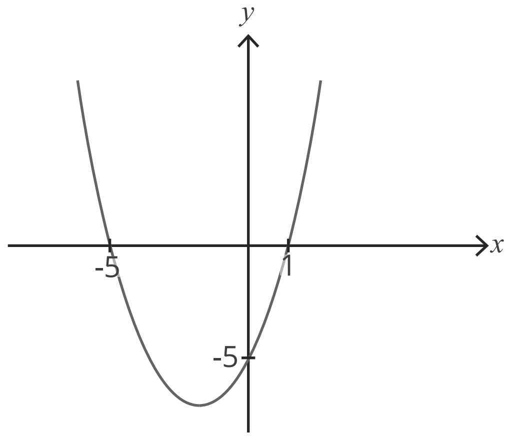
   $0 = x^2 + 4x - 5$

   $0 = (x - 1)(x + 5)$

   $x = 1$, $x = -5$

   (1, 0) and (-5, 0)

c. Hence, sketch the graph of $y = x^2 + 4x - 5$, clearly indicating any points of intersection with the axes.



2. Consider the curve with equation $y = x^2 + 8x - 1$.

a. Find the coordinates of the turning point of this curve.

$$x^2 + 8x - 1 = (x + 4)^2 - 4^2 - 1$$

$$= (x + 4)^2 - 17$$

The coordinates of the turning point are (-4, -17)

b. State whether the turning point is a maximum or minimum. Justify your answer.

This must be a minimum since the coefficient of x^2 is positive. It is u-shaped and so the turning point must be the lowest point of the curve.

3. Sketch the graph of $y = x^2 + 4x - 21$, clearly indicating any points of intersection with the axes and the location of the turning point of the curve.

$y = 0^2 + 4 \times 0 - 21 = -21$

$0 = x^2 + 4x - 21$

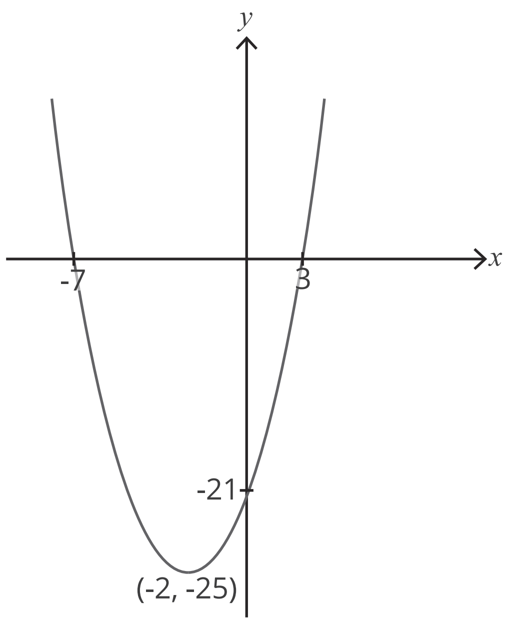
$0 = (x + 7)(x - 3)$

$x = -7, x = 3$

$x^2 + 4x - 21 = (x + 2)^2 - 2^2 - 21$

$= (x + 2)^2 - 25$

The turning point has coordinates (-2, -25)



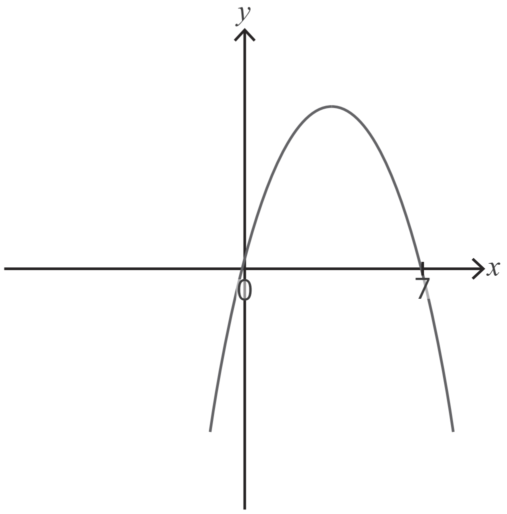
4. Sketch the graph of $y = -x^2 + 7x$, clearly indicating any points of intersection with the axes.

$y = -0^2 + 7 \times 0 = 0$

$0 = -x^2 + 7x$

$0 = -x(x - 7)$

$x = 0$ or $x = 7$



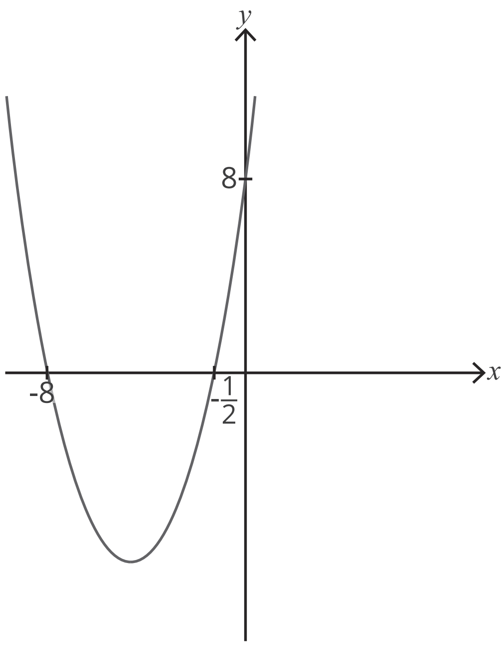
5. Sketch the graph of $y = 2x^2 + 17x + 8$, clearly indicating any points of intersection with the axes.

$y = 2 \times 0^2 + 17 \times 0 + 8 = 8$

$0 = 2x^2 + 17x + 8$

$0 = (2x + 1)(x + 8)$

$x = -\frac{1}{2}$, $x = -8$



Quadratic Equations and Inequalities (Answers)

1. Solve by factorising:

   a. $x^2 - 11x + 28 = 0$

   $(x - 7)(x - 4) = 0$

   $x = 7, x = 4$

   c. $y^2 + 4y - 35 = 2y$

   $y^2 + 2y - 35 = 0$

   $(y - 5)(y + 7) = 0$

   $y = 5, y = -7$

   b. $3x^2 - 16x - 12 = 0$

   $(3x + 2)(x - 6) = 0$

   $x = -\frac{2}{3}, x = 6$

   d. $6n^2 - 6n = -n^2$

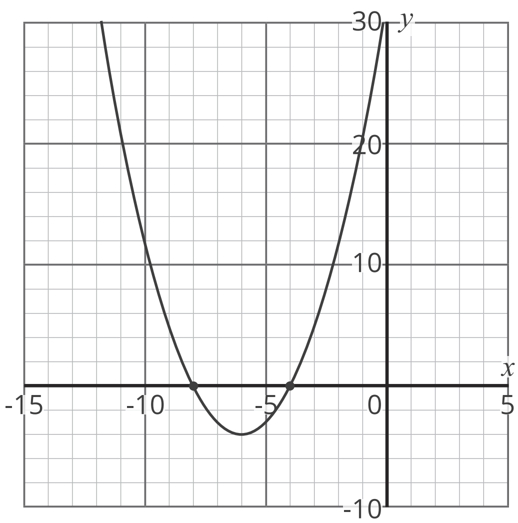
   $7n^2 - 6n = 0$

   $n(7n - 6) = 0$

   $n = 0, n = \frac{6}{7}$

2. Solve the inequalities by sketching the graph:

   a. $x^2 + 12x + 32 \geq 0$



$(x + 8)(x + 4) = 0$

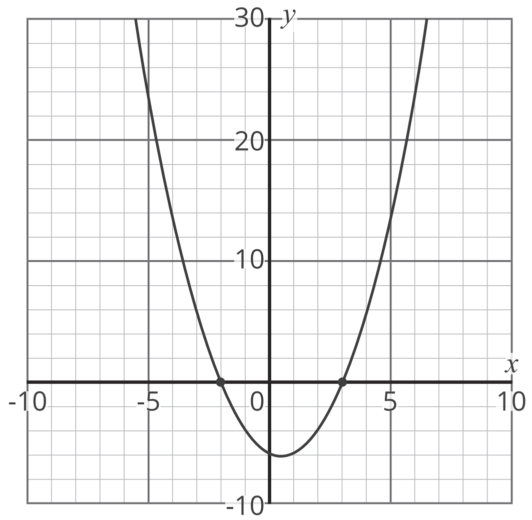
$x = -8, x = -4$

The graph of $y = x^2 + 12x + 32$

is shown as:

$x \leq -8, x \geq -4$

b. $x^2 - 2x - 8 < -x - 2$



$x^2 - x - 6 < 0$

$(x - 3)(x + 2) = 0$

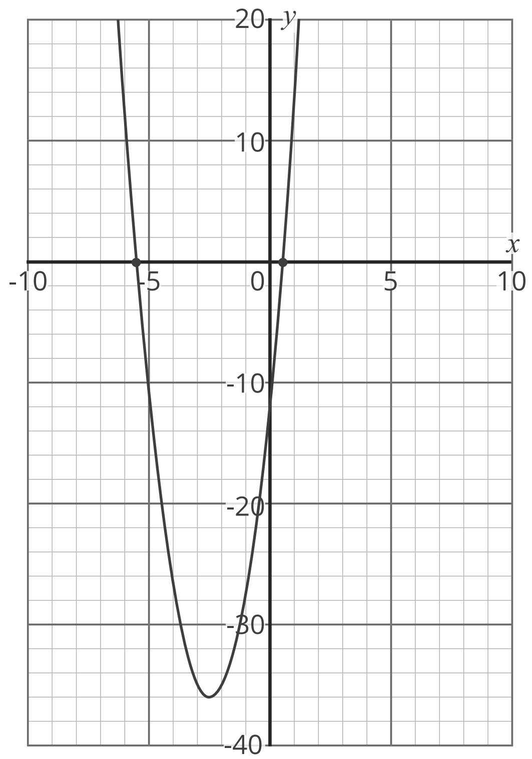
$x = 3$, $x = -2$

The graph of $y = x^2 - x - 6$

is shown as:

$-2 < x < 3$

c. $4x^2 + 20x - 11 \leq 0$



$(2x - 1)(2x + 11) = 0$

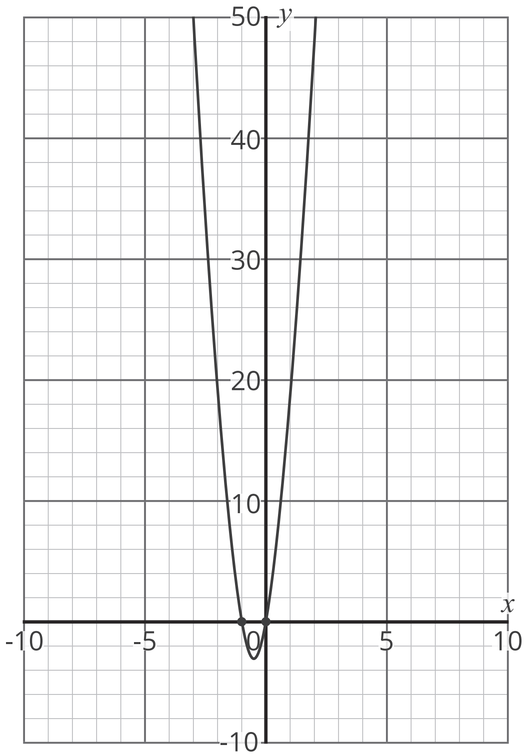
$x = \frac{1}{2}$, $x = -\frac{11}{2}$

The graph of $y = 4x^2 + 20x - 11$

is shown as:

$-\frac{11}{2} \leq x \leq \frac{1}{2}$

d. $8x^2 + 9x > 2x$



$8x^2 + 7x > 0$

$x(8x + 7) = 0$

$x = 0, x = -\frac{7}{8}$

The graph of $y = 8x^2 + 7x$

is shown as:

$x < -\frac{7}{8}, x > 0$

3. Solve by completing the square, writing surds in their simplest form:

a. $x^2 + 8x + 3 = 0$

$(x + 4)^2 - 13 = 0$

$(x + 4)^2 = 13$

$x + 4 = \pm\sqrt{13}$

$x = -4 \pm \sqrt{13}$

b. $x^2 + 3x - 7 = 0$

$(x + \frac{3}{2})^2 - \frac{37}{4} = 0$

$(x + \frac{3}{2})^2 = \frac{37}{4}$

$x + \frac{3}{2} = \pm\sqrt{\frac{37}{4}}$

$x = -\frac{3}{2} \pm \sqrt{\frac{37}{4}}$

$x = -\frac{3}{2} \pm \frac{\sqrt{37}}{2}$

c. $2x^2 + 12x + 1 = 0$

$2(x^2 + 6x) + 1 = 0$

$2((x + 3)^2 - 9) + 1 = 0$

$2(x + 3)^2 - 17 = 0$

$2(x + 3)^2 = 17$

$(x + 3)^2 = \frac{17}{2}$

$x + 3 = \pm\sqrt{\frac{17}{2}}$

$x = -3 \pm \sqrt{\frac{17}{2}}$

d. $x^2 + 16x + 3 = -9$

$x^2 + 16x + 12 = 0$

$(x + 8)^2 - 52 = 0$

$(x + 8)^2 = 52$

$x + 8 = \pm\sqrt{52}$

$x = -8 \pm \sqrt{52}$

$x = -8 \pm 2\sqrt{13}$

The Quadratic Formula (Answers)

1. Solve $2x^2 + 5x + 1 = 0$, giving your answers correct to 3 significant figures.

   $$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times 1}}{2 \times 2}$$

   $x = -2.28$, $x = -0.219$

2. Solve $5x^2 + 2x = 19$, giving your answers correct to 3 significant figures.

   $5x^2 + 2x - 19 = 0$

   $$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 5 \times (-19)}}{2 \times 5}$$

   $x = -2.16$, $x = 1.76$

3. Explain why the graph of the equation $y = x^2 + 4x + 9$ does not intersect the x-axis.
   Use the discriminant $\Delta = b^2 - 4ac$

   $\Delta = 4^2 - 4 \times 1 \times 9$

   $\Delta = -20$

   $\Delta < 0$

   Since the discriminant is less than zero, there are no real solutions to the equation $x^2 + 4x + 9 = 0$ and so the graph does not intersect the x-axis.

Simultaneous Equations (Answers)

1. Solve each pair of simultaneous equations:

a. $x + y = 14$

$2x - y = 16$

Add the equations:

$\mathbf{3x = 30}$

$\mathbf{x = 10}$

$\mathbf{10 + y = 14}$

$\mathbf{y = 4}$

$\mathbf{x = 10, y = 4}$

b. $3x + 2y = -4$

$2x + y = -3$

Multiply the second equation by 2, then subtract the equations:

$\mathbf{4x + 2y = -6}$

$\mathbf{x = -2}$

$\mathbf{2 \times (-2) + y = -3}$

$\mathbf{-4 + y = -3}$

$\mathbf{y = 1}$

$\mathbf{x = -2, y = 1}$

c. $2x + 5y = 26$

$y = x + 1$

Substitute the second equation into the first. You can use elimination, but you will need to rearrange first.

$\mathbf{2x + 5(x + 1) = 26}$

$\mathbf{7x + 5 = 26}$

$\mathbf{7x = 21}$

$\mathbf{x = 3}$

$\mathbf{y = 3 + 1}$

$\mathbf{y = 4}$

$\mathbf{x = 3, y = 4}$

d. $x^2 + y^2 = 10$

$y = x + 2$

Substitute the second equation into the first.

$\mathbf{x^2 + (x + 2)^2 = 10}$

$\mathbf{2x^2 + 4x + 4 = 10}$

$\mathbf{2x^2 + 4x - 6 = 0}$

$\mathbf{x^2 + 2x - 3 = 0}$

$\mathbf{(x + 3)(x - 1) = 0}$

$x = -3$ or $x = 1$

$y = -3 + 2$ or $y = 1 + 2$

$y = -1$ or 3

$x = -3$ and $y = -1$, $x = 1$ and $y = 3$

e. $2x^2 = y^2 - 8$

$y - x = 2$

Rearrange and then substitute the second equation into the first.

$y = x + 2$

$2x^2 = (x + 2)^2 - 8$

$2x^2 = x^2 + 4x + 4 - 8$

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$x = 2$

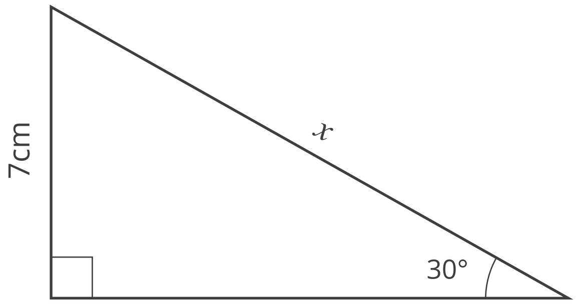
$y = 2 + 2 = 4$

$x = 2, y = 4$

Right-angled Trigonometry (Answers)

1. In each question, find the value of x. Give your answers to 3s.f.

a.



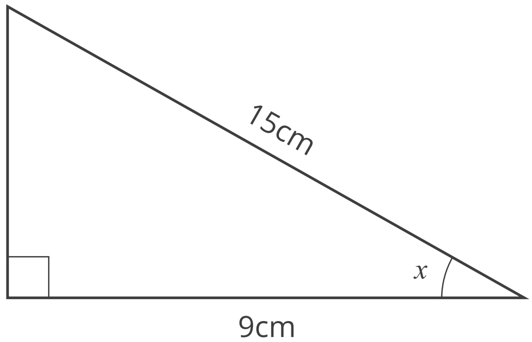
$\sin\theta = \frac{O}{H}$

$H = \frac{O}{\sin\theta}$

$x = \frac{7}{\sin(30°)}$

x = **14.0cm**

b.



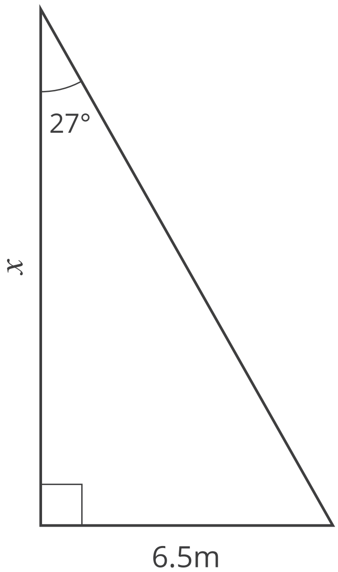
$\cos\theta = \frac{A}{H}$

$\theta = \cos^{-1}\left(\frac{A}{H}\right)$

$x = \cos^{-1}\left(\frac{9}{15}\right)$

x = **53.1°**

c.



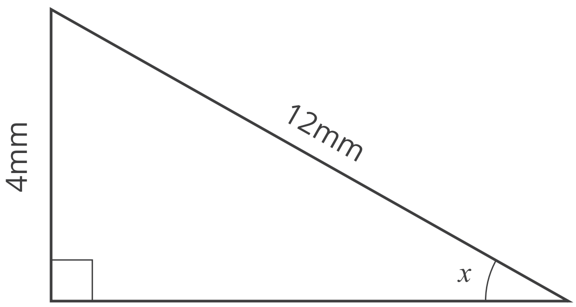
$\tan\theta = \frac{O}{A}$

$A = \frac{O}{\tan\theta}$

$x = \frac{6.5}{\tan(27°)}$

x = **12.8m**

d.



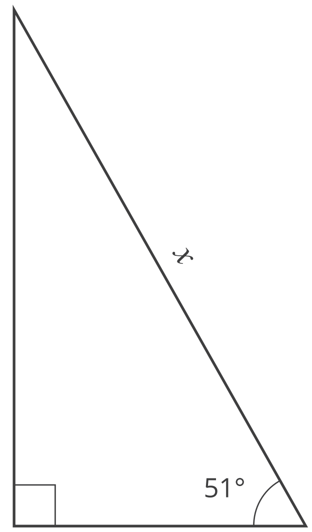
$\sin\theta = \frac{O}{H}$

$\theta = \sin^{-1}\left(\frac{O}{H}\right)$

$x = \sin^{-1}\left(\frac{4}{12}\right)$

x = **19.5°**

e.



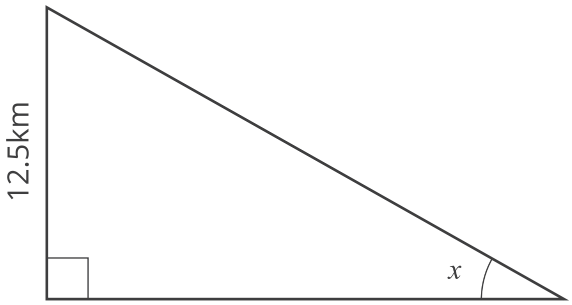
$\cos\theta = \frac{A}{H}$

$H = \frac{A}{\cos\theta}$

$x = \frac{9.2}{\cos(51°)}$

$x = 14.6$m

f.



$\tan\theta = \frac{O}{A}$

$\theta = \tan^{-1}\left(\frac{O}{A}\right)$

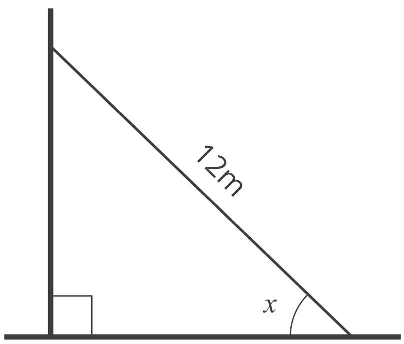
$x = \tan^{-1}\left(\frac{12.5}{15}\right)$

$x = 39.8°$

2. Zara is an engineer. She is building a wall, which needs to be supported as it is built. The ground is perfectly horizontal, the wall is perfectly vertical and the support is a straight steel beam. For the wall to be safe, the angle between the beam and the ground must be less than 55°.

   If the support beam is 12m long and the end of the support is 3.5m away from the base of the wall, is the wall safely supported?

In a worded question, start by drawing the situation:

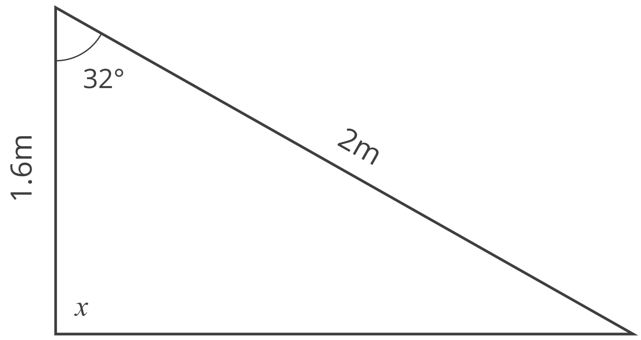


$\cos\theta = \frac{A}{H}$

$x = \cos^{-1}\left(\frac{3.5}{12}\right) = 73.042...°$

73.0° > 55° so the wall is not safely supported.

3. Look at the diagram below. Is x a right angle? You must justify your answer.



Right-angled trigonometry only works on right-angled triangles. Therefore, if the values satisfy one of the trigonometric ratios, x is 90°.

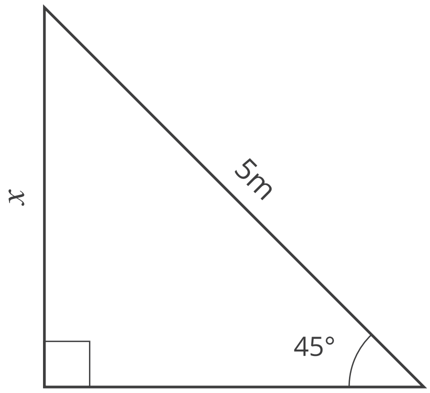
Assuming the triangle is right-angled, the hypotenuse is 2m and the adjacent side is 1.6m, with respect to the angle marked as 32°. This means, in a right-angled triangle, the top angle (which we will call y) can be found by:

$\cos y = \frac{1.6}{2}$

$y = \cos^{-1}(0.8) = 36.9°$ **(to 3s.f.)**

The angle given in our diagram is not 36.9° so this triangle is not right-angled.

4. Find the length of side x, giving your answer as a surd in its simplest form.



$x = \sin(45°) \times 5$

$x = \frac{5}{\sqrt{2}}$

$x = \frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$x = \frac{5\sqrt{2}}{2}$ m

5. A mnemonic that can help to remember the trigonometric ratios is: **S**ydney **o**pera **h**ouse **c**an **a**lways **h**old **t**housands **o**f **A**ustralians. Try to make up your own mnemonic.

Example answers:

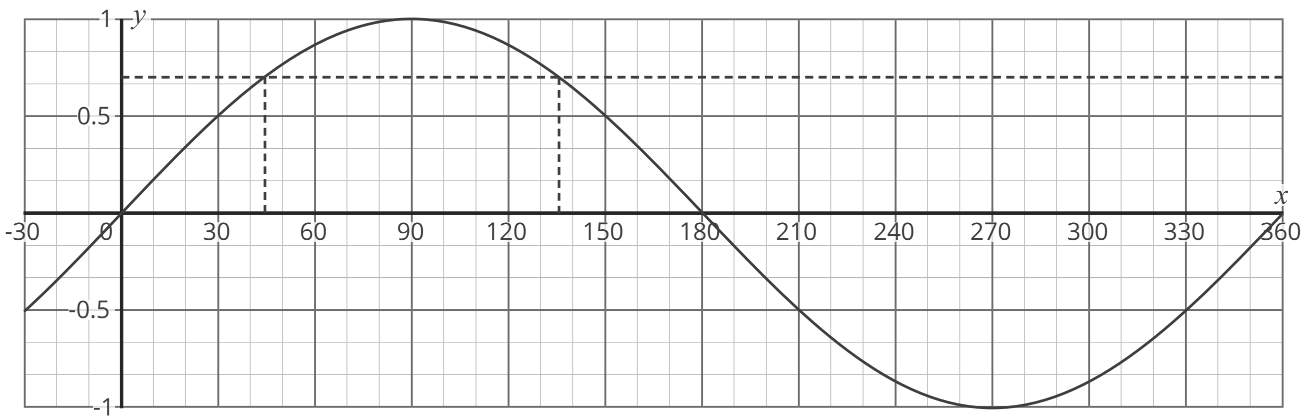
Some Old Hippies Can't Always Hide Their Old Age

Sense Of Humour Can Always Help To Overcome Awkwardness

Summer On Holiday, Christmas At Home, Teachers Off Anyway

Trigonometric Graphs (Answers)

1. x is an obtuse angle such that $\sin x = 0.7$. What is x, to 1d.p.?



$\sin^{-1}(0.7) = 44.4°$

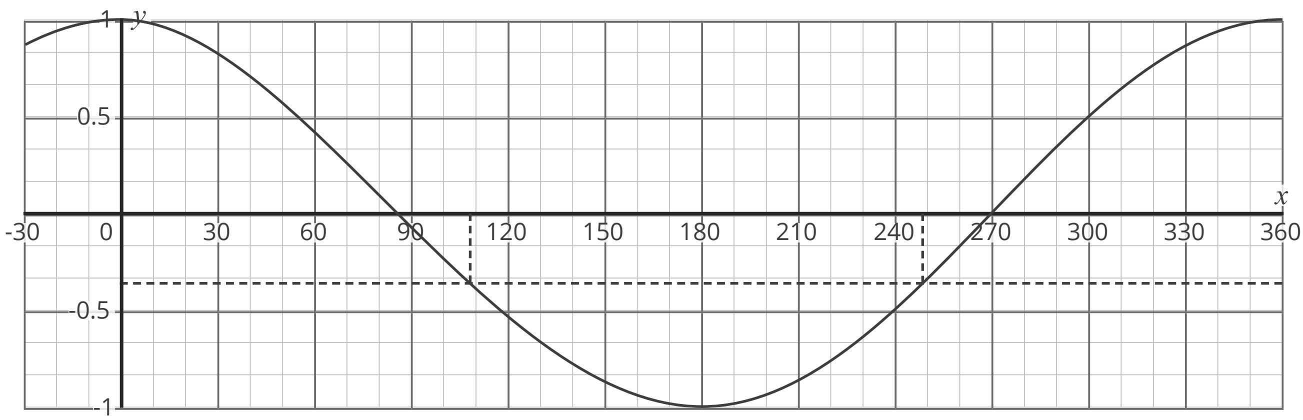
but x is an obtuse angle so $90° < x < 180°$.

Look at the graph of $y = \sin x$

We can use the symmetry of the graph about 90° to calculate the actual value of x.

$180 – 44.4 = 135.6°$

2. x is a reflex angle such that $\cos x = -0.3$. What is x, to 3s.f.?



$\cos^{-1}(-0.3) = 107°$

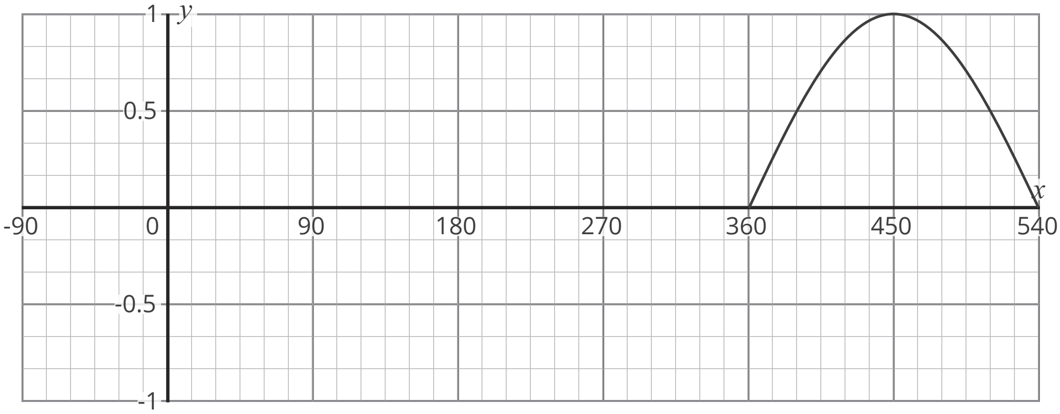
but x is a reflex angle so $180° < x < 360°$.

Look at the graph of $y = \cos x$

We can use the symmetry of the graph about 180° to calculate the actual value of x.

$360 – 107 = 253°$

3. Sketch the graph of $y = \sin x$, where $360° \leq x \leq 540°$.



4. For each set of values of x, how many solutions are there for $\sin x = 0.2$?
   a. $0° \leq x \leq 360°$

      2

   b. $-360° \leq x \leq 360°$

      4

   c. $-180° \leq x \leq 180°$

      2

5. For each set of values of x, how many solutions are there for $\cos x = -0.7$?
   a. $0° \leq x \leq 720°$

      4

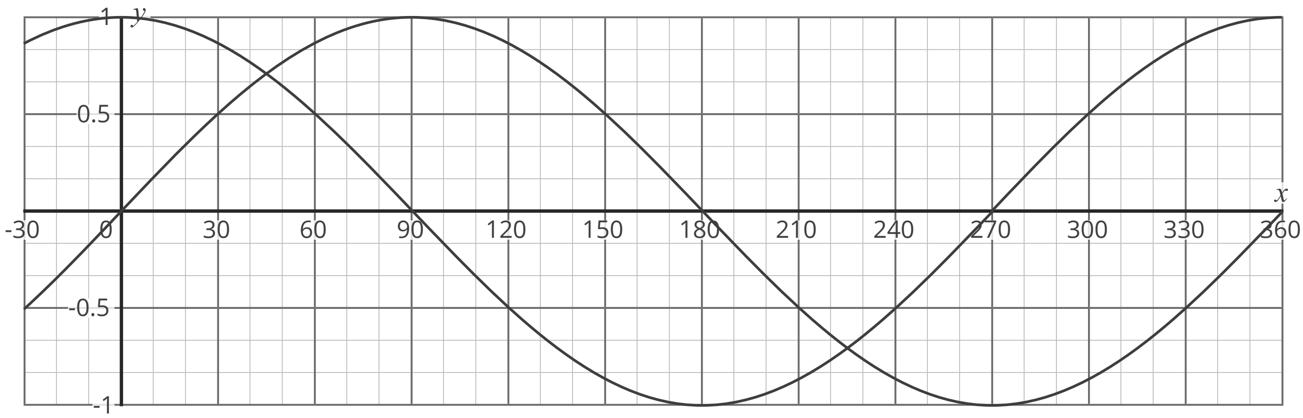
   b. $-360° \leq x \leq 0°$

      2

   c. $0° \leq x \leq 180°$

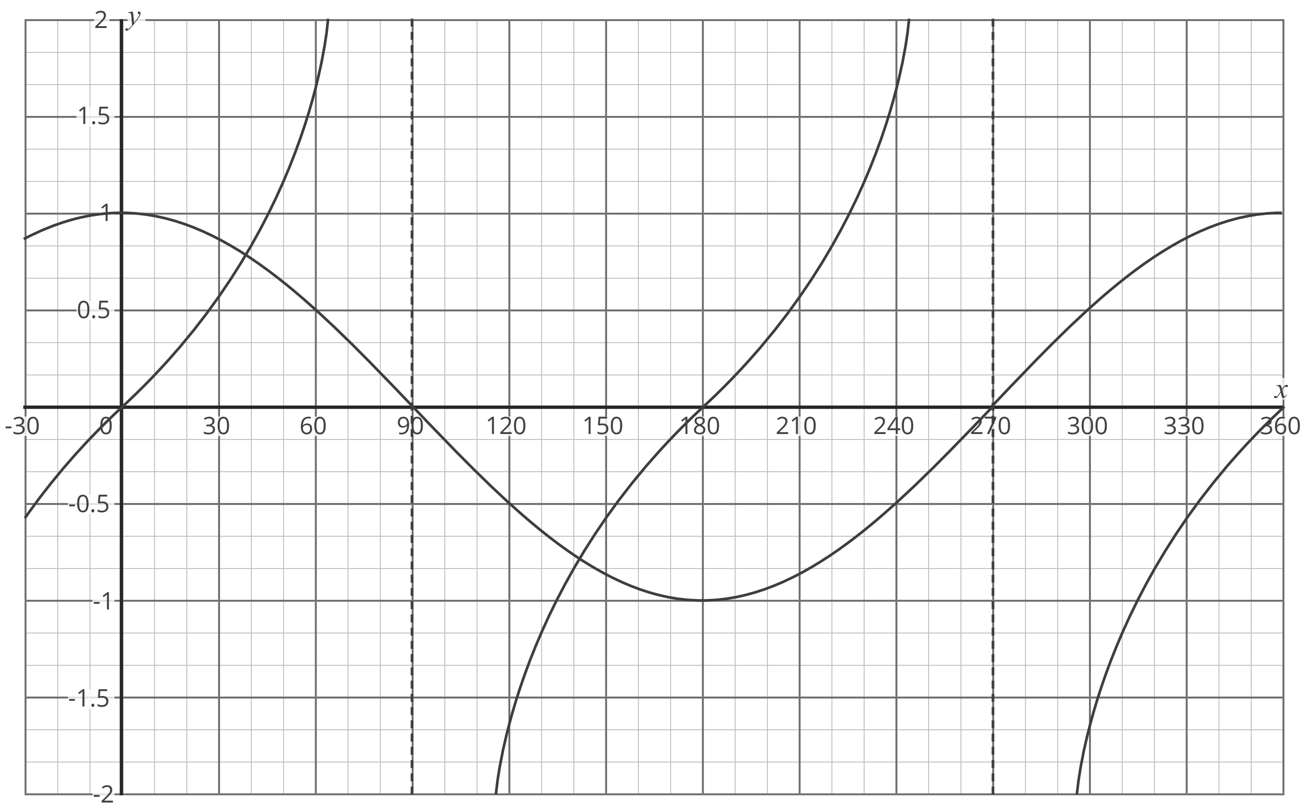
      1

6. On the same set of axes, plot y = sin$x$ and y = cos$x$ for 0° ≤ x ≤ 360°. Use your graph to find the values of x for which cos$x$ = sin$x$



x = 45° and 225°

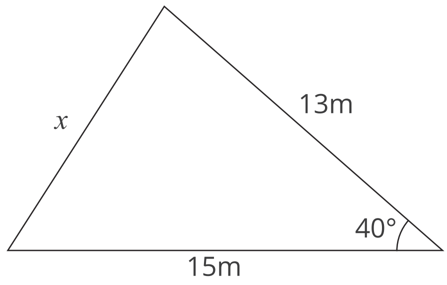
7. Plot y = tan$x$ and y = cos$x$ on the same set of axes, for 0° ≤ x ≤ 360°. Write down the x value for each of the asymptotes for y = tan$x$. What is the value of y = cos$x$ for each of these values?



Asymptotes are at x = 90° and x = 270°. At these values, cos$x$ = 0.

Sine and Cosine Rules (Answers)

1. Find the length of side x, giving your answer correct to 1d.p.



Using the cosine rule:

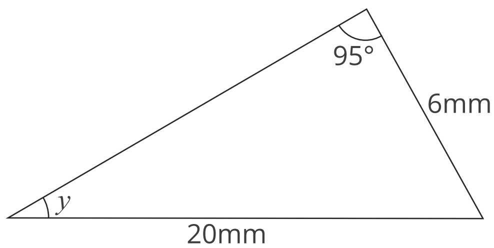
$a^2 = b^2 + c^2 - 2bc\cos A$

$x^2 = 13^2 + 15^2 - 2 \times 13 \times 15 \times \cos(40°)$

$x^2 = 95.24...$

$x = 9.8$**m (1d.p.)**

2. Find the size of angle y, giving your answer correct to 3s.f.



Using the sine rule:

$\frac{\sin A}{a} = \frac{\sin B}{b}$

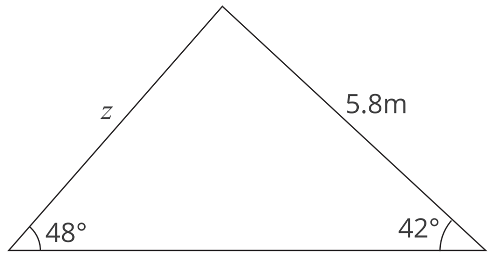
$\frac{\sin(95°)}{20} = \frac{\sin y}{6}$

$\sin y = \frac{6 \times \sin(95°)}{20}$

$\sin y = 0.298...$

$y = 17.4°$ **(3s.f.)**

3. Find the length of side z, giving your answer correct to 1d.p.



Using the sine rule:

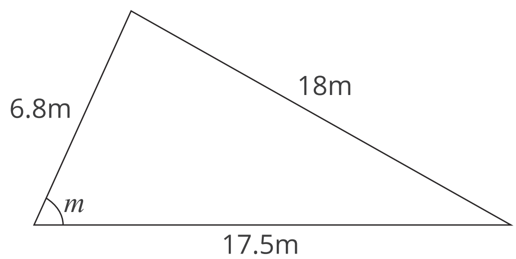
$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\frac{z}{\sin(42°)} = \frac{5.8}{\sin(48°)}$

$z = \frac{\sin(42°) \times 5.8}{\sin(48°)}$

$z = 5.2$**m (1d.p.)**

4. Find the size of angle m. Give your answer correct to 3s.f.



Using the cosine rule for angles:

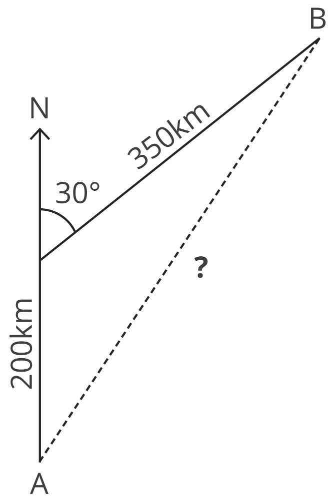
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos m = \frac{6.8^2 + 17.5^2 - 18^2}{2 \times 6.8 \times 17.5}$

$\cos m = 0.119...$

$m = 83.1°$ **(3s.f.)**

5. A plane takes off from Airport A, flies 200km due north, then turns on a bearing of 030° and flies a further 350km before landing at Airport B. How far is airport A from airport B in a straight line? Give your answer correct to the nearest kilometre.



Always start by sketching worded questions. We are looking for the length of the side labelled "?".

Using the cosine rule:

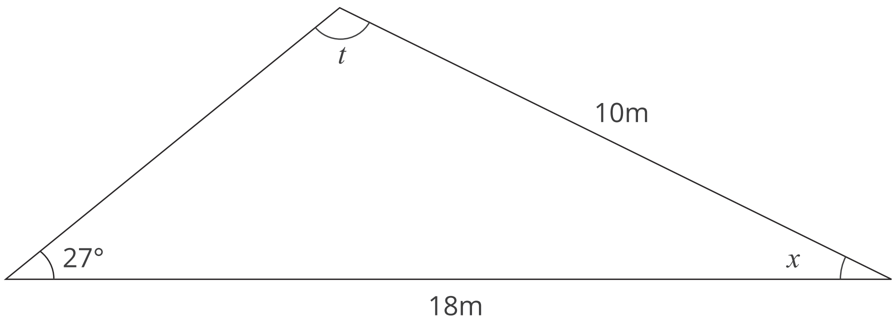
$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 200^2 + 350^2 - 2 \times 200 \times 350 \times \cos(150°)$

$a^2 = 283\ 743.55...$

$a = 533\text{km}$

6. Given that angle t is obtuse, find the size of angle x, giving your answer correct to 1d.p.



Use the sine rule to find angle t.

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin(27°)}{10} = \frac{\sin t}{18}$

$\sin t = \frac{18 \times \sin(27°)}{10}$

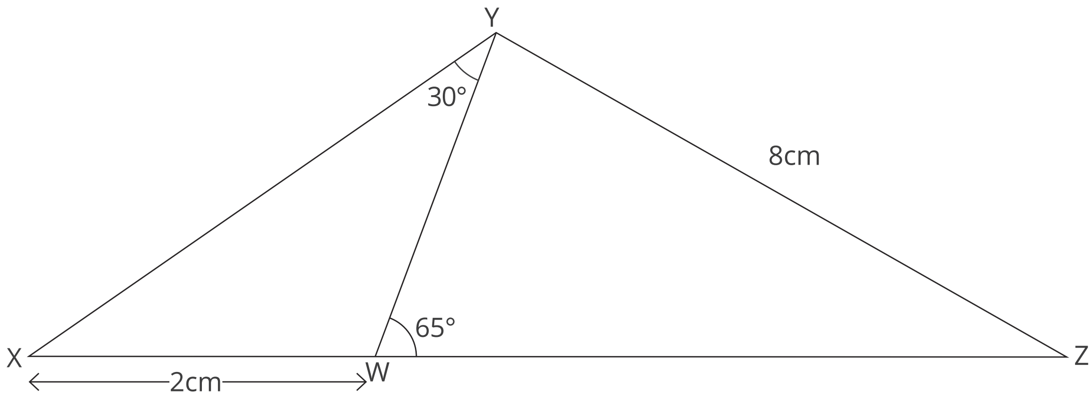
$\sin t = 0.817...$

$t = 125.1...°$ (Your calculator will give 54.8°... Use the sine graph to find an answer > 90°.)

$x = 180 - 27 - 125.2$ (Angles in a triangle add up to 180°.)

$x = 27.8°$ (1d.p.)

7. A triangle XYZ is divided by a line YW. Side YZ is 8cm, angle XYW is 30° and angle YWZ is 65°. Find the size of angle WZY, giving your answer correct to 3s.f.



Angle XWY = 115° (Angles on a straight line add up to 180°.)

Angle YXW = 35° (Angles in a triangle add up to 180°.)

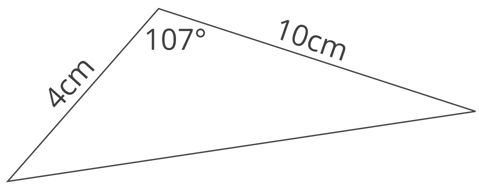
Using the sine rule on triangle WXY gives side YW = 2.294...cm

Using the sine rule on triangle YWZ gives angle WZY = 15.1° (3s.f.)

Area of Any Triangle (Answers)

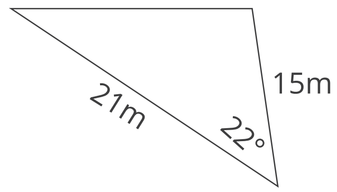
1. Find the area of each of these triangles.

a.



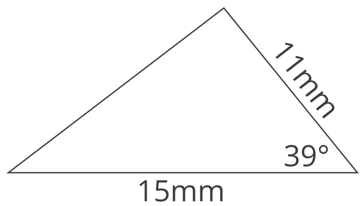
area = $\frac{1}{2} \times 4 \times 10 \times \sin(107°) = 19.1\text{cm}^2$

b.



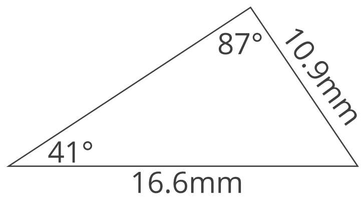
area = $\frac{1}{2} \times 21 \times 15 \times \sin(22°) = 59.0\text{m}^2$

c.



area = $\frac{1}{2} \times 11 \times 15 \times \sin(39°) = 51.9\text{mm}^2$

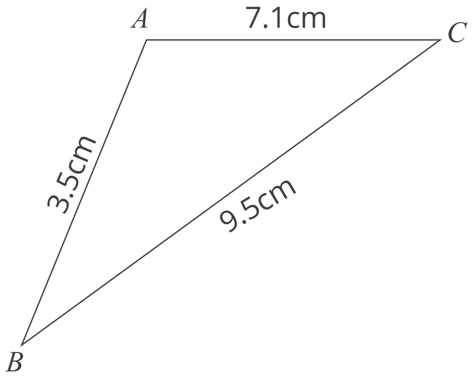
d.



$C = 180 - (41 + 87) = 52°$

area = $\frac{1}{2} \times 16.6 \times 10.9 \times \sin(52°) = 71.3\text{mm}^2$

2. By finding angle C, find the area of the triangle ABC.



Using the cosine rule:

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

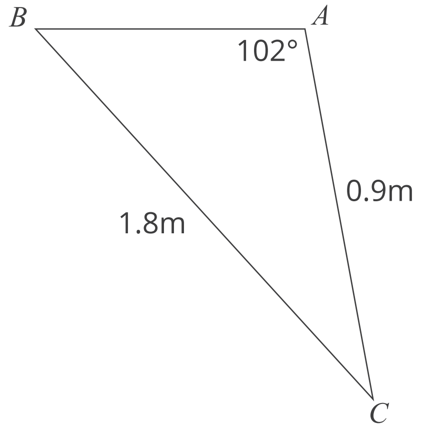
$\cos C = \frac{9.5^2 + 7.1^2 - 3.5^2}{2 \times 9.5 \times 7.1} = 0.951...$

$C = \cos^{-1}(0.951) = 17.8...°$

Area = $\frac{1}{2} \times 7.1 \times 9.5 \times \sin(17.8°)$

$= 10.3\text{cm}^2$

3. Find the area of the triangle below.



Using the sine rule:

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin(102°)}{1.8} = \frac{\sin B}{0.9}$

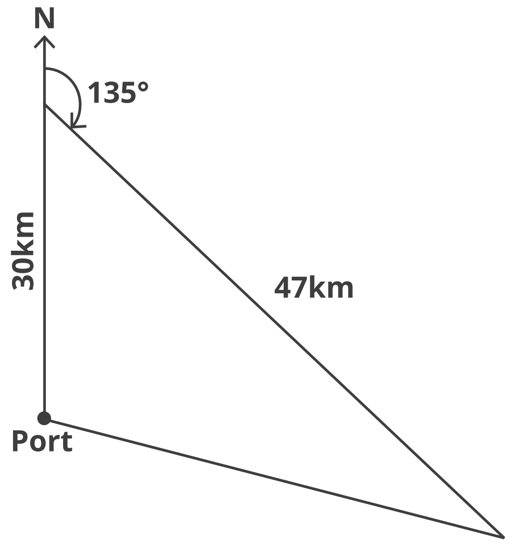
$\sin B = \frac{0.9\sin(102°)}{1.8} = 0.489...$

$B = \sin^{-1}(0.489) = 29.2...°$

Angle $C = 180 - (102 + 29.2...) = 48.7...°$

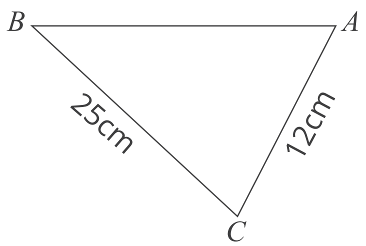
Area = $\frac{1}{2} \times 1.8 \times 0.9 \times \sin(48.7°) = 0.609\text{m}^2$

4. A ship patrols a dangerous area in the North Sea. The ship starts in port and travels 30km due North. It then travels 47km on a bearing on 135 before returning to port. Find the area enclosed by the ships journey.



Area = $\frac{1}{2}$ × 47 × 30 × sin(45°) = 499km^2

5. The area of the triangle below is 138cm^2. Find the size of angle C.



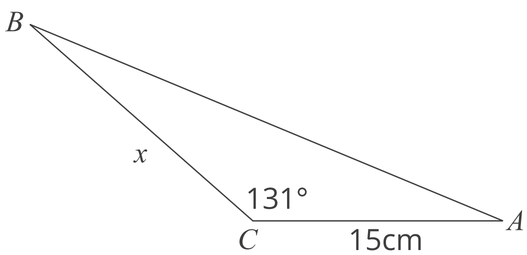
$138 = \frac{1}{2} \times 25 \times 12 \times \sin C$

$138 = 150\sin C$

$\sin C = \frac{138}{150} = 0.92...$

$C = \sin^{-1}(0.92) = 66.9°$

6. The area of the triangle below is 152cm^2. Find the length of the side x.

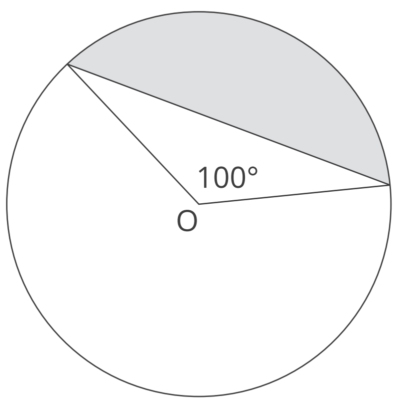


$152 = \frac{1}{2} \times 15 \times x \times \sin(131°)$

$152 = 5.66x$

$x = 26.9$cm

7. The shape below is a circle centre O and radius 4.5cm. Find the area of the shaded area below.



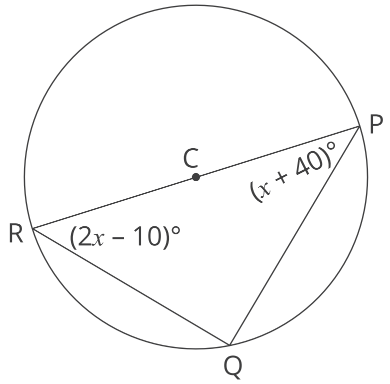
Area of triangle =
$\frac{1}{2}$ × 4.5 × 4.5 × sin(100°) = 9.97cm^2

Area of segment =
$\frac{100}{360} \times 4.5^2 \times \pi$ = 17.6 cm^2

Shaded area = 17.6 – 9.97 = 7.70cm^2

Circle Theorems (Answers)

1. The points P, Q and R lie on the circumference of a circle, centre C. Find the value of x.



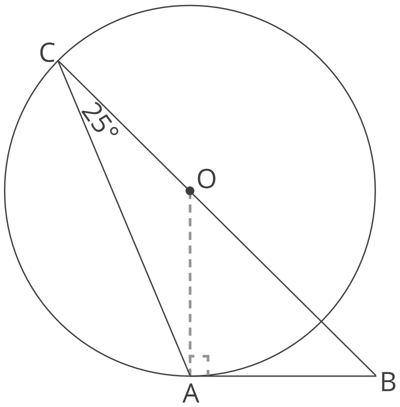
$2x - 10 + x + 40 + 90 = 180$

$3x + 120 = 180$

$3x = 60$

$x = 20$

2. Below is a circle, centre O and radius 4cm. Given that AB is a tangent to the circle at point A, find the angle ABC.



CO and AO are radiuses, so they are both 4cm.

AOC is an isosceles triangle:

CAO = 25°

COA = 180 – (25 + 25) = 130°

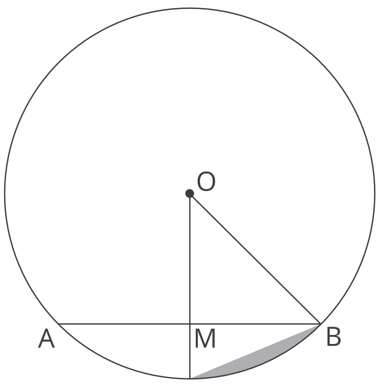
COB is a straight line:

AOB = 180 – 130 = 50°

OA will be perpendicular to AB:

OBA = 180 – (90 + 50) = 40°

3. The circle below has centre O and radius 10cm. Points A and B lie on the circumference, such that AB is 13.5cm long and OMB is a right angle.



a. Find the angle OBM.

MB = $\frac{1}{2}$ × 13.5 = 6.75cm

OBM = $\cos^{-1}(\frac{6.75}{10})$ = 47.5° (1d.p.)

b. Find the area of the shaded region.

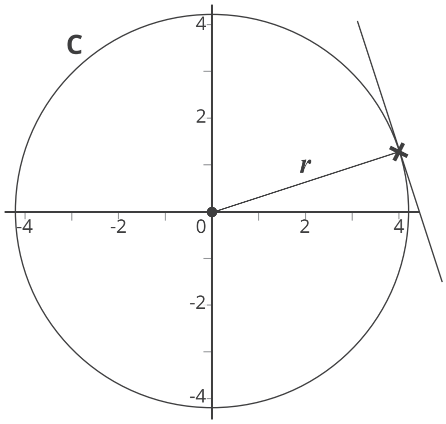
Angle MOB = 180 – (90 + 47.5...)
= 42.4...°

Area of segment = $\frac{42.4...}{360} \times \pi \times 10^2$
= 37.0...cm^2

Area of triangle = $\frac{1}{2}$ × 10 × 10 × sin(42.4...°)
= 33.75cm^2

Shaded area = 37.0... – 33.75 = 3.3cm^2 (1d.p.)

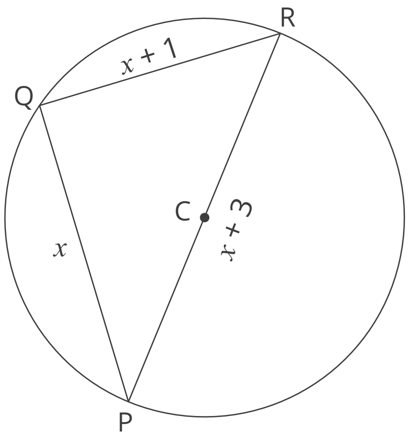
4. A circle, C, has centre (0, 0) and passes through the point (4, 1). Find the gradient of the tangent to C at the point (4, 1).



gradient = $\frac{1-0}{4-0} = \frac{1}{4}$

The radius has gradient $\frac{1}{4}$. The radius and tangent are perpendicular, so the tangent will have gradient -4 as -4 × $\frac{1}{4}$ = -1.

5. The points P, Q and R lie on the circumference of a circle, centre C. Find the exact area of the circle below. All measurements are given in centimetres.



$$(x + 1)^2 + x^2 = (x + 3)^2$$

$$x^2 + 2x + 1 + x^2 = x + 6x + 9$$

$$x^2 - 4x - 8 = 0$$

$x = 2 + 2\sqrt{3}$ (as x is a length, we take the positive solution)

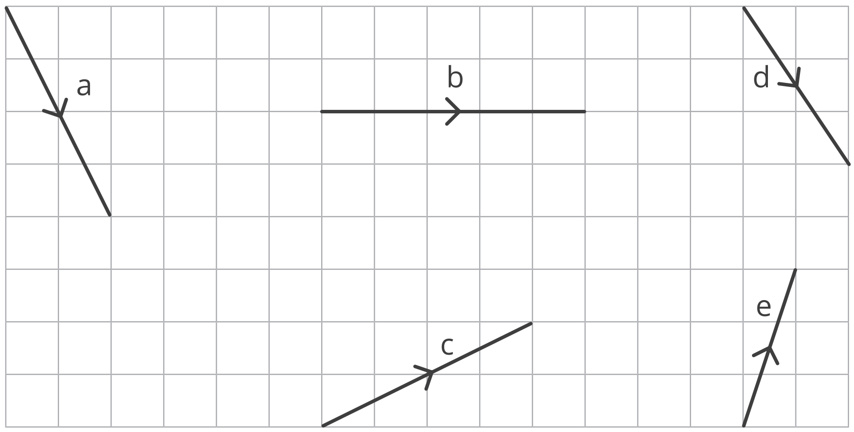
$$\begin{aligned} \text{diameter} &= x + 3 \\ &= 2 + 2\sqrt{3} + 3 \\ &= (5 + 2\sqrt{3})\text{cm} \end{aligned}$$

$$\text{radius} = \frac{5 + 2\sqrt{3}}{2}\ \text{cm}$$

$$\begin{aligned} \text{Area} &= \pi \times \left(\frac{5 + 2\sqrt{3}}{2}\right)^2 \\ &= \pi \times \frac{25 + 20\sqrt{3} + 12}{4} \\ &= \frac{(37 + 20\sqrt{3})\pi}{4}\ \text{cm}^2 \end{aligned}$$

Vectors (Answers)

1. Write each of the vectors below as a column vector.



a. $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

b. $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

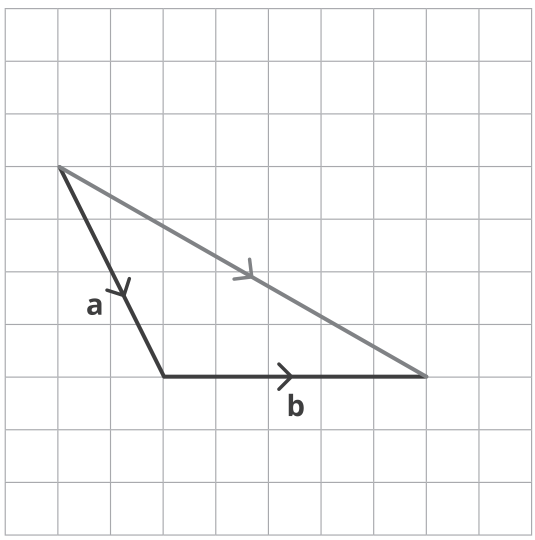
c. $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

d. $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

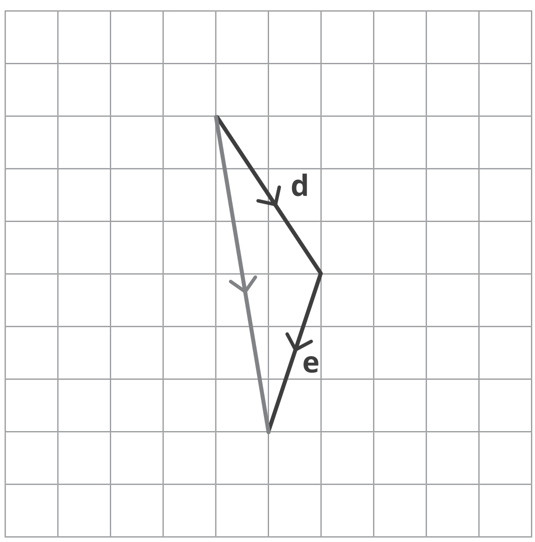
e. $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2. Using the vectors in question 1, use diagrams to show each of these vector additions. You do not need to give the resultant vector.

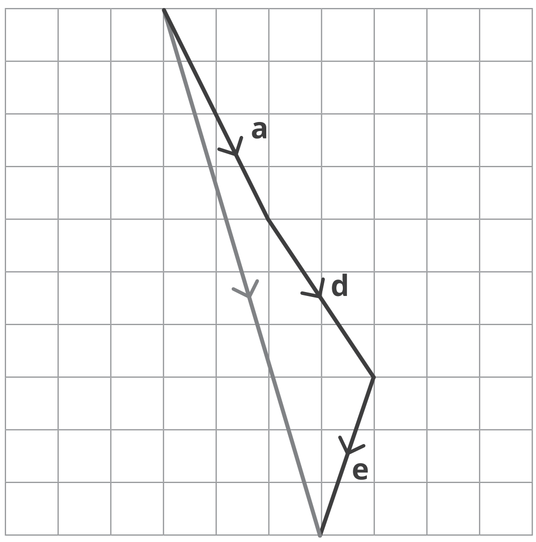
a. **a** + **b**



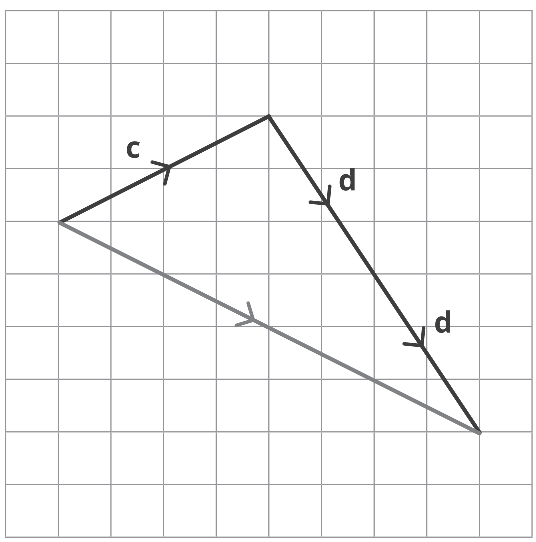
b. **d** – **e**



c. **a** + **d** – **e**



d. **c** + 2**d**



3. For the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, write each of the following as column vectors.

a. $\mathbf{a} + \mathbf{b}$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b. $\mathbf{a} - \mathbf{c}$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

c. $\mathbf{a} - \mathbf{b} + \mathbf{c}$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

d. $2\mathbf{a}$

$2\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

e. $3\mathbf{a} - \mathbf{c}$

$3\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ -7 \end{pmatrix}$

f. $\mathbf{c} - \frac{1}{2}\mathbf{b}$

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

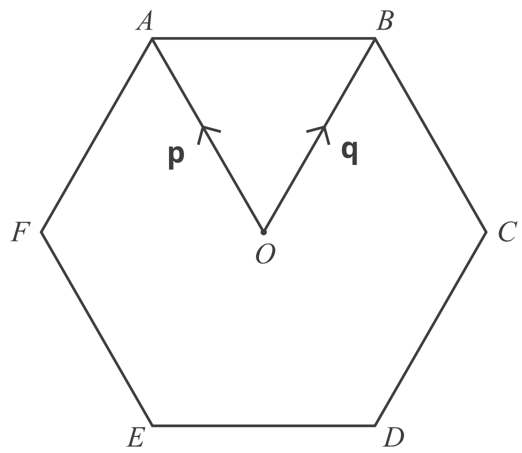
g. $2\mathbf{a} - 3\mathbf{c}$

$2\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ -14 \end{pmatrix}$

h. $\frac{1}{2}\mathbf{a} - 2\mathbf{b} + \frac{1}{4}\mathbf{c}$

$\frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ -\frac{7}{2} \end{pmatrix}$

This diagram is used for questions 4 and 6.



4. The diagram above shows a regular hexagon, centre O. Find the vectors below in terms of **p** and **q**.

a. $\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{q} - \mathbf{p}$

b. $\overrightarrow{ED}$

$\overrightarrow{ED} = \overrightarrow{AB} = \mathbf{q} - \mathbf{p}$

c. $\overrightarrow{OE}$

$\overrightarrow{OE} = \overrightarrow{BO} = -\mathbf{q}$

d. $\overrightarrow{DA}$

$\overrightarrow{DA} = 2\overrightarrow{OA} = 2\mathbf{p}$

e. $\overrightarrow{AF}$

$\overrightarrow{AF} = \overrightarrow{BO} = -\mathbf{q}$

f. $\overrightarrow{BF}$

$\overrightarrow{BF} = \overrightarrow{BA} + \overrightarrow{AF} = -\overrightarrow{AB} + \overrightarrow{AF}$
$= -(\mathbf{q} - \mathbf{p}) - \mathbf{q} = \mathbf{p} - 2\mathbf{q}$

5. Given that M is the midpoint of BC, find the vector $\overrightarrow{EM}$ in terms of **p** and **q**.

$\overrightarrow{BC} = \overrightarrow{AO} = -\mathbf{p}$

$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = -\frac{1}{2}\mathbf{p}$

$\overrightarrow{EB} = 2\overrightarrow{OB} = 2\mathbf{q}$

$\overrightarrow{EM} = \overrightarrow{EB} + \overrightarrow{BM} = 2\mathbf{q} - \frac{1}{2}\mathbf{p}$

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